Theorem 6. *The functions*

$$F(x, z) = h^{-1}[h(x)c(z)], \qquad G(x, y) = h^{-1}[h(x) + h(y)]$$

with arbitrary continuous c and arbitrary continuous strictly increasing h such that $h(e) = 0$, $h(\epsilon) = 1$ constitute the general solution of

$$F[G(x, y), z] = G[F(x, z), F(y, z)]$$

$$(x, y, z, F(x, z), G(x, y), F(y, z), F[G(x, y), z], G[F(x, z), F(y, z)] \in [e, \epsilon])$$

if $F(x, y)$ is bounded from below and $G(x, y)$ is assumed to be continuous, increasing, and associative for $e \leqslant x, y < \epsilon$, and with e as identity element:

$$G(e, x) = G(x, e) = x.$$

7.1.4. AN APPLICATION TO THE FOUNDATIONS OF PROBABILITY THEORY.[100] Generalizing Kolmogorov's system of axioms for probability theory, A. RÉNYI 1954, 1962, has suggested that probability theory be based on the following system of axioms for the *conditional probability*, $p(A|V)$ (probability of A under the supposition V). Here $A \in \mathscr{U}$ and $V \in \mathscr{V}$ where $\mathscr{U}$ is a *σ-algebra of subsets of a set M* (that is, *$\mathscr{U}$ is not empty; if $A \in \mathscr{U}$, then also the complementary subset $M - A \in \mathscr{U}$; if $A_n \in \mathscr{U}$ $(n = 1, 2, \ldots)$ then the sum $\Sigma A_n \in \mathscr{U}$ too*) and $\mathscr{V} \subseteq \mathscr{U}$. We use the usual expressions and notations of the algebras of events; that is, the events (subsets) $A + B$, AB, $\mathbf{0}$ have the following meanings:

$A + B$: the event A *or* the event B (sum of the subsets A and B),

AB: the event A *and* the event B (product of the subsets A and B),

O: the *impossible* event (the empty set).

[100] See among others S. N. BERNSTEIN 1917, 1934; R. T. COX 1946, 1961; G. A. BARNARD 1949, 1951; I. J. GOOD 1950; H. RICHTER 1952[a, b, c], 1953, 1954, 1956; J. ACZÉL 1955[b], 1961[f], 1963[a]; L. JÁNOSSY 1955, 1960; E. RUFENER 1959, 1963; N. N. VOROBYEV 1961. Some of these papers generalize the system of axioms of A. N. KOLMOGOROV, "Grundbegriffe der Wahrscheinlichkeitsrechnung" (*Ergeb. Math. Grenzgebiete* [2]3), Berlin 1933, in the same sense as we have generalized that of A. RÉNYI 1954, 1962, at this point. Naturally, the considerations following here can also be used in particular to generalize Kolmogorov's system of axioms.

The axioms of Rényi in a form more suitable for our purposes and not materially changed are

$$0 = p(O|V) \leqslant p(A|V), \tag{1}$$

$$p(V|V) = p(W|W) = 1, \tag{2}$$

$$p[(AV)|V] = p(A|V), \tag{3}$$

$$p[(AB)|V] = p[A|(BV)]p(B|V), \tag{4}$$

$$p[(A + B)|V] = p(A|V) + p(B|V), \tag{5}$$

where in (5) it is assumed that A, B are mutually exclusive events, $(AB = O)$. The last axiom suffices if only a finite number of mutually exclusive alternative events are considered; in the infinite case, in place of (5) we would assume

$$p\left[\left(\sum_{k=1}^{\infty} A_k\right)\middle| V\right] = \sum_{k=1}^{\infty} p(A_k|V) \qquad (\text{if } A_i A_j = O \text{ for } i \neq j),$$

which, however, we shall not need here.

It is our aim to generalize the system of axioms (1) through (5) in an obvious way, by assuming in place of (5) and (4) that $p[(A + B)|V]$ depends only on $p(A|V)$ and $p(B|V)$, whereas $p(AB|V)$, on the other hand, depends only on $p(A|BV)$ and on $p(B|V)$. In other words, we replace in (4) and (5) the special functions xy and $x + y$ with arbitrary functions:

$$p(AB|V) = F_{\mathbf{V}}[p(A|BV), p(B|V)], \tag{6}$$

$$p[(A + B)|V] = G_{\mathbf{V}}[p(A|V), p(B|V)]. \tag{7}$$

The notation already shows that the functions may depend on assumption V. Then, for example,

$$p[(A + B)|CV] = G_{\mathbf{CV}}[p(A|CV), p(B|CV)].$$

Here, too, we assume

$$p[(AV)|V] = p(A|V), \tag{3}$$

as well as the conditions

$$e = p(O|V) \leqslant p(A|V), \tag{8}$$

$$p(V|V) = (pW|W) = \epsilon > e, \tag{9}$$

which slightly generalize Eqs. (1) and (2). Let functions $F_{\mathbf{V}}(x, y)$, $G_{\mathbf{V}}(x, y)$ be defined for $x, y \in [e, \epsilon]$. Furthermore, we assume that $G_{\mathbf{V}}(x, y)$ *increases continuously* with x and y, which is fulfilled in a trivial manner in the case of $G_V(x, y) = x + y$. In other words, $p[(A + B)|V]$ increases continuously with $p(A|V)$ and with $p(B|V)$.

We prove that these assumptions are necessary and sufficient for the possibility of introducing in a unique and continuous manner a new probability that satisfies the original axioms (1) through (5). More precisely, we prove the following

Theorem 1. *Assumptions*

$$e = p(O|V) \leqslant p(A|V), \tag{8}$$

$$p(V|V) = p(W|W) = \epsilon > e, \tag{9}$$

$$p[(AV)|V] = p(A|V), \tag{3}$$

$$p(AB|V) = F_V[p(A|BV), p(B|V)], \tag{6}$$

$$p[(A + B)|V] = G_V[p(A|V), p(B|V], \tag{7}$$

as well as the continuous increase of $G_V(x, y)$ *are necessary and sufficient for the existence of a continuous and strictly increasing function* $h(t)$ *such that* $h(e) = 0$, $h(\epsilon) = 1$, *and*

$$\bar{p}(A|V) = h[p(A|V)] \tag{10}$$

satisfies conditions (1), (2), (3), (4), *and* (5), *if* $p(A|V)$ *takes all values between* e *and* ϵ *where* $A \in \mathscr{U}$(*a* σ*-algebra of subsets of a set* M) and $\mathscr{V} \subseteq \mathscr{U}$ *is connected in the sense that for two arbitrary* $V \in \mathscr{V}$, $W \in \mathscr{V}$ *there exist events* $C_1, C_2, \ldots, C_n \in \mathscr{V}$ *such that* $C_1V \in \mathscr{V}$, $C_2C_1 \in \mathscr{V}, \ldots, WC_n \in \mathscr{V}$.

For proof, we use the following laws of formal logic (of the algebra of events):

$$VV = V,$$
$$(AV)A = AV,$$
$$O + A = A,$$
$$(A + B) + C = A + (B + C),$$
$$(A + B)C = AC + BC,$$

from which follow the relations

$$p(A|VV) = p(A|V),$$
$$p[(AV)A|V] = p(AV|V),$$
$$p[(O + A)|V] = p(A|V),$$
$$p\{[(A + B) + C]|V\} = p\{[A + (B + C)]|V\},$$
$$p[(A + B)C|V] = p[(AC + BC)|V].$$

Because of (6), (7), and (3), as well as (8) and (9), we conclude from these the following properties of the functions $F_{\mathbf{V}}(x, y)$, $G_{\mathbf{V}}(x, y)$:

$$\begin{aligned}
F_{\mathbf{V}}(x, \epsilon) &= F_{\mathbf{V}}[p(A|V), p(V|V)] = F_{\mathbf{V}}[p(A|VV), p(V|V)] \\
&= p(AV|V) = p(A|V) = x, \\
F_{\mathbf{V}}(\epsilon, x) &= F_{\mathbf{V}}[p(V|V), p(A|V)] = F_{\mathbf{V}}\{p(AV|AV), p(A|V)\} \\
&= p[(AV)A|V] = p(AV|V) = p(A|V) = x, \\
G_{\mathbf{V}}(e, x) &= G_{\mathbf{V}}[p(O|V), p(A|V)] = p[(O + A)|V)] = p(A|V) = x, \\
G_{\mathbf{V}}[G_{\mathbf{V}}(x, y), z] &= p\{[(A + B) + C]|V\} = p\{[A + (B + C)]|V\} \\
&= G_{\mathbf{V}}[x, G_{\mathbf{V}}(y, z)], \\
F_{\mathbf{V}}[G_{\mathbf{CV}}(u, v), z] &= F_{\mathbf{V}}\{p[(A + B)|CV], p(C|V)\} = p[(A + B)C|V] \\
&= p[(AC + BC)|V] = G_V[p(AC|V), p(BC|V)] \\
&= G_{\mathbf{V}}[F_{\mathbf{V}}(u, z), F_{\mathbf{V}}(v, z)],
\end{aligned}$$

where we set

$$x = p(A|V), \quad y = p(B|V), \quad z = p(C|V), \quad u = p(AC|V), \quad v = p(BC|V).$$

If we simply write

$$F_{\mathbf{V}}(x, y) = F(x, y),$$

$$G_{\mathbf{V}}(x, y) = H(x, y), \tag{11}$$

$$G_{\mathbf{CV}}(x, y) = G(x, y), \tag{12}$$

we obtain for these functions the following conditions:

$$F(x, y) \geqslant e, \tag{13}$$

$$F(x, \epsilon) = F(\epsilon, x) = x, \tag{14}$$

$$H(e, x) = x, \tag{15}$$

$$H[H(x, y), z) = H[x, H(y, z)], \tag{16}$$

$$F[G(u, v), z] = H[F(u, z), F(v, z)]. \tag{17}$$

Equation (17) is the special case $K = J = F$ of 7.1.3(4); (16) is the associativity equation for H; Eqs. (15) and (14) indicate that the operations H, F have identity elements, whereas (13) asserts the boundedness of the function F from below. On the other hand, G *was supposed continuous increasing in* x *and* y. If in (17) we set $z = \epsilon$, then because of (14)

$$G(u, v) = H(u, v), \tag{18}$$

so that (15) and (16) demonstrate the existence of an identity element e and the associativity of G:

$$G(e, x) = x,$$

$$G[G(x, y), z] = G[x, G(y, z)],$$

and (17) becomes the distributivity equation 7.1.3(20)

$$F(G(x, y), z] = G[F(x, z), F(y, z)].$$

Equations (11), (12), and (18) imply

$$G_{\mathbf{V}}(x, y) = G_{\mathbf{CV}}(x, y),$$

and by interchanging C and V

$$G_{\mathbf{C}}(x, y) = G_{\mathbf{CV}}(x, y) = G_{\mathbf{VC}}(x, y) = G_{\mathbf{V}}(x, y);$$

and by the connectedness of $\mathscr{V}$, there exist for any two $V \in \mathscr{V}$, $W \in \mathscr{V}$ events $C_1, C_2, \ldots, C_n \in \mathscr{V}$ such that $C_1 V \in \mathscr{V}$, $C_2 C_1 \in \mathscr{V}, \ldots, WC_n \in \mathscr{V}$, and thus

$$\begin{aligned} G_{\mathbf{V}}(x, y) &= G_{\mathbf{C}_1\mathbf{V}}(x, y) = G_{\mathbf{C}_1}(x, y) = G_{\mathbf{C}_2\mathbf{C}_1}(x, y) = G_{\mathbf{C}_2}(x, y) = \cdots \\ &= G_{\mathbf{C}_n}(x, y) = G_{\mathbf{WC}_n}(x, y) = G_{\mathbf{W}}(x, y) \end{aligned}$$

that is, $G_{\mathbf{V}}(x, y)$ *is independent of* V.

According to Sect. 7.1.3, Theorem 6, we find that

$$H(x, y) = G(x, y) = h^{-1}[h(x) + h(y)], \tag{19}$$

$$F(x, z) = h^{-1}[h(x)c(z)], \tag{20}$$

$$h(e) = 0, \qquad h(\epsilon) = 1. \tag{21}$$

Because of (14) and (21), however,

$$z = F(\epsilon, z) = h^{-1}[h(\epsilon)c(z)] = h^{-1}[c(z)],$$

thus

$$c(z) = h(z),$$

so finally (19) and (20) go over into

$$F(x, z) = h^{-1}[h(x)h(z)],$$
$$H(x, y) = G(x, y) = h^{-1}[h(x) + h(y)].$$

These functions do in fact satisfy our conditions. If we substitute them into (6), (7), (8), (9), and (3), we obtain—cf. also (21)—for

$$\bar{p}(A|V) = h[p(A|V)], \tag{10}$$

where h is continuous and strictly monotonically increasing,

$$\begin{aligned} \bar{p}(AB|V) &= \bar{p}(A|BV)\bar{p}(B|V), \\ \bar{p}[(A + B)|V] &= \bar{p}(A|V) + \bar{p}(B|V), \\ 0 &= \bar{p}(O|V) \leqslant \bar{p}(A|V), \\ \bar{p}(V|V) &= \bar{p}(W|W) = 1, \\ \bar{p}(AV|V) &= \bar{p}(AV), \end{aligned}$$

that is, (1) through (5). The inverse statement is trivial. Thus we have proved our Theorem 1.

It is also easy to show the following

Theorem 2. *If $\mathscr{V}$ is not connected in the sense of Theorem* 1, *but splits into connected parts $\mathscr{V}_1$, $\mathscr{V}_2$, ..., the other suppositions of Theorem* 1 *remaining valid, then in the assertion of Theorem* 1, *h is replaced by h_1, h_2, ..., for which $h_i(e) = 0$, $h_i(\epsilon) = 1$* ($i = 1, 2, \ldots$); and in place of (10)

$$\bar{p}(A|V) = h_i[p(A|V)], \text{ for } V \in \mathscr{V}_i \qquad (i = 1, 2, \ldots)$$

holds.

7.2. Reduction to Partial Differential Equations and Functional Differential Equations

7.2.1. Euler's Differential Equation of Homogeneous Functions. Related Equations and Generalizations. Euler 1755 (see also, for example, S. Gołąb 1932; H. H. Downing and S. J. Jasper 1948; H. A. Thurston 1960) obtained from the functional equation

$$F(xz, yz) = F(x, y)z^k \qquad (z > 0) \tag{1}$$

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