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On the Present Status of the Radiation Problem

by A. Einstein

This Journal recently published contributions by Messrs. H. A. Lorentz[1], Jeans[2], and Ritz[3] which help to simplify the critical interpretation of the present status of this extremely important problem. In the belief that it would be useful if all those who have seriously thought about this matter communicate their views, even if they have not been able to arrive at a final result, I join in with the following contribution.

1. The simplest form in which we can express the currently understood laws of electrodynamics is the set of Maxwell-Lorentz partial differential equations. I regard the equations containing retarded functions, in contrast to Mr. Ritz,[3] as merely auxiliary mathematical forms. The reason I see myself compelled to take this view is first of all that those forms do not subsume the energy principle, while I believe that we should adhere to the strict validity of the energy principle until we have found important reasons for renouncing this guiding star. It is certainly correct that Maxwell's equations for empty space, taken by themselves, do not say anything, they only represent an intermediary construct. But, as is well known, exactly the same could be said about Newton's equations of motion, as well as about any theory that needs to be supplemented by other theories in order to yield a picture of a complex of phenomena. What distinguishes the Maxwell-Lorentz differential equations from the forms that contain retarded functions is the fact that they yield an expression for the energy and the momentum of the system under consideration for any instant of time, relative to any unaccelerated coordinate system. With a theory that operates with retarded forces it is not possible to describe the instantaneous state of a system at all without using earlier states of the system for this description. For example, if a light source A had emitted a light complex toward the screen B, but it has not yet reached the screen B, then, according to theories operating with retarded forces, the light complex is represented by nothing except the processes that have taken place in the emitting body during the preceding emission. Energy and momentum—if one does not want to renounce these quantities altogether—must then be represented as time integrals.

Mr. Ritz certainly claims that experience forces us to abandon these differential equations and introduce the retarded potentials. However, his arguments do not seem sound to me.

If one defines, in agreement with Ritz,

$$f_1 = \frac{1}{4\pi} \int \frac{\varphi\left(x', y', z', t - \frac{r}{c}\right)}{r} dx', dy', dz'$$

and

$$f_2 = \frac{1}{4\pi} \int \frac{\varphi\left(x', y', z', t + \frac{r}{c}\right)}{r} dx', dy', dz',$$

then both f_1 and f_2 are solutions of the equation

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \Delta f = \varphi(x\ y\ z\ t),$$

and hence

$$f_3 = a_1 f_1 + a_2 f_2$$

[1] H. A. Lorentz, *Phys. Zeit.* **9,** 562-563, 1908.
[2] J. H. Jeans, *Phys. Zeit.* **9,** 853-855, 1908.
[3] W. Ritz, *Phys. Zeit.* **9,** 903-907, 1908.

is also a solution if $a_1 + a_2 = 1$. But it is not true that the solution f_3 is a more general solution than f_1 and that one specializes the theory by setting $a_1 = 1$, $a_2 = 0$. Setting

$$f(x, y, z, t) = f_1$$

amounts to calculating the electromagnetic effect at the point x, y, z from those motions and configurations of the electric quantities that took place *prior to* the time point t. Setting

$$f(x, y, z, t) = f_2,$$

one determines the electromagnetic effects from the motions and configurations that take place *after* the time point t.

In the first case the electric field is calculated from the totality of the processes producing it, and in the second case from the totality of the processes absorbing it. If the whole process occurs in a (finite) space bounded on all sides, then it can be represented in the form

$$f = f_1$$

as well as in the form

$$f = f_2.$$

If we consider a field that is emitted from the finite into the infinite, we can, naturally, only use the form

$$f = f_1,$$

exactly because the totality of the absorbing processes *is not taken into consideration.* But here we are dealing with a misleading paradox of the infinite. Both kinds of representation can always be used, regardless of how distant the absorbing bodies are imagined to be. Thus, one cannot conclude that the solution $f = f_1$ is more special than the solution $a_1 f_1 + a_2 f_2$, where $a_1 + a_2 = 1$.

That a body does not "receive energy from infinity unless some other body loses a corresponding quantity of energy" cannot be brought up as an argument either, in my opinion. First of all, if we want to stick to experience, we cannot speak of infinity but only of spaces lying outside the space considered. Furthermore, it is no more permissible to infer irreversibility of the electromagnetic elementary processes from the non-observability of such a process than it is permissible to infer irreversibility of the elementary processes of atomic motion from the second law of thermodynamics.

2. Jeans' interpretation can he disputed on the grounds that it might not be permissible to apply the general results of statistical mechanics to cavities filled with radiation. However, the law deduced by Jeans can also be arrived at in the following way[1].

According to Maxwell's theory, an ion capable of oscillating about an equilibrium position in the direction of the X-axis will, on average, emit and absorb equal amounts of energy per unit time only if the following relation holds between the mean oscillation energy $\overline{E}_\nu$ and the energy density of the radiation ϱ_ν at the proper frequency ν of the oscillator:

$$\overline{E}_\nu = \frac{c^3}{8\pi\nu^2}\varrho_\nu, \qquad \text{(I)}$$

where c denotes the speed of light. If the oscillating ion can also interact with gas molecules (or, generally, with a system that can be described by means of the molecular theory), then we must necessarily have, according to the statistical theory of heat,

$$\overline{E}_\nu = \frac{RT}{N} \qquad \text{(II)}$$

(R = gas constant, N = number of atoms in one gram-atom, T = absolute temperature), if, on average, no energy is transferred by the oscillator from the gas to the radiation space[2]

From these two equations we arrive at

$$\varrho_\nu = \frac{R}{N}\frac{8\pi}{c^3}\nu^2 T, \qquad \text{(III)}$$

i.e., exactly the same law that has also been found by Messrs. Jeans and H. A. Lorentz[3]

3. There can be no doubt, in my opinion, that our current theoretical views necessarily lead to the law

[1] Cf. A. Einstein, *Ann. d. Phys.* (4) **17,** 133-136, 1905.

[2] M. Planck, *Ann. d. Phys.* **1,** 69, 1900. (ed: original publication has 99 instead of 69 as pg. ref. There is no Planck paper with pg. ref. 99. The intended reference is M. Planck, *Ann. d. Phys.* **1,** 69-122, 1900.) *Vorlesungen über die Theorie der Wärmestrahlung.* III. Kapitel. (ed: III. Kapitel refers to "Dritter Abschnitt.")

[3] It should be explicitly noted that this equation is an irrefutable consequence of the statistical theory of heat. The attempt, on p. 178 of the book by Planck just cited, to question the general validity of Equation II, is based—it seems to me—only on a gap in Boltzmann's considerations, which has been filled in the meantime by Gibbs' investigations.

propounded by Mr. Jeans. However, we can consider it as almost equally well established that formula (III) is not compatible with the facts. Why, after all, do solids emit visible light only above a fixed, rather sharply defined temperature? Why are ultraviolet rays not swarming everywhere if they are indeed constantly being produced at ordinary temperatures? How is it possible to store highly sensitive photographic plates in cassettes for a longtime if they constantly produce short-wave rays? For further arguments I refer to §166 of Planck's repeatedly cited work. Thus, we must say that experience forces us to reject either equation (I), required by electromagnetic theory, or equation (II), required by statistical mechanics, or both equations.

4. We must ask ourselves how Planck's radiation theory relates to the theory which is indicated in **2.** and which is based on our currently accepted theoretical foundations. In my opinion the answer to this question is made harder by the fact that Planck's presentation of his own theory suffers from a certain logical imperfection. I will now try to explain this briefly.

a) If one adopts the standpoint that the irreversibility of the processes in nature is only *apparent,* and that the irreversible process consists in a transition to a more probable state, then one must first give a definition of the probability W of a state. The only definition worthy of consideration, in my opinion, would be the following:

Let $A_1, A_2 \dots A_l$ be all the states that a closed system at a certain energy content can assume, or, more accurately, all the states that we can distinguish in such a system with the help of certain auxiliary means. According to the classical theory, after a certain time the system will assume one particular state (e.g., A_l) and then remain in this state (thermodynamic equilibrium). However, according to statistical theory the system will keep assuming, in an irregular sequence, all these states $A_1 \dots A_l$[1]. If the system is observed over a very long time period Θ, there will be a certain portion τ_ν of this time such that during τ_ν and during τ_ν only, the system occupies the state A_ν. The quantity $\overline{\tau}_\nu/\Theta$ will have a definite limiting value, which we call the probability W that the system has assumed state A_ν.

Proceeding from this definition, one can show that the entropy must satisfy the equation

$$S = \frac{R}{N} \log W + \text{const},$$

where the constant is the same for all states of the same energy.

b) Neither Mr. Boltzmann nor Mr. Planck have given a definition of W.

They set, purely formally, $W =$ number of complexions of the state under consideration.

If one now demands that these complexions be equally probable, where the probability of the complexion is defined in the same way that we have defined the probability of the state under a), one will obtain precisely the definition for the probability of a state given under a); however, the logically unnecessary element "complexion" has been used in the definition.

Even though the indicated relation between S and W is valid only if the probability of a complexion is defined in the manner indicated, or in a manner equivalent to it, neither Mr. Boltzmann nor Mr. Planck has defined the probability of a complexion. But Mr. Boltzmann did clearly realize that the molecular-theoretical picture he had chosen dictated his choice of complexions in a quite definite manner; he discussed this on pgs. 404 and 405 of his paper "Über die Beziehung..."[2] that appeared in the *Wiener Sitzungsberichten* of 1877[3]. Similarly, Mr. Planck would have had no freedom in the choice of complexions in the resonator theory of radiation. He could have been permitted to postulate the pair of equations

$$S = \frac{R}{N} \log W$$

and

$$W = \text{number of complexions}$$

only if he had appended the condition that the complexions must be chosen such that, in the statistically-based theoretical model chosen by him, they had been found to be equally probable. In this way he would have arrived at the formula advocated by Jeans. Though every physicist must rejoice that Mr. Planck disregarded these requirements in such a fortunate manner, it should not

[1] That only this last interpretation is tenable follows immediately from the properties of Brownian motion.

[2] ed: This reference appears to be to the publication: L. Boltzmann, "Über die Beziehung eines allgemeinen mechanischen Satzes zum zweiten Hauptsatz der Wärmetheorie." Wien 1877, in: Sitzungberichte der mathematisch-naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften, 76: 373-435. Reprinted in Boltzmann, *Wissenschftliche Abhandlungen,* Leipzig, 1909, 3 vols, Vol. 2, pgs. 164-223.

[3] Cf. also L. Boltzmann, Vorlesungen über Gastheorie (Lectures on Gas Theory), Vol. I, p. 40, lines 9-23.

be forgotten that the Planck radiation formula is incompatible with the theoretical foundation from which Mr. Planck started out.

5. It is simple to see how one could modify the foundations of the Planck theory in order to have the Planck radiation formula truly result from the theoretical foundations. I will not present the pertinent derivations here but will rather just refer to my papers on this subject[1]. The result is as follows: One arrives at the Planck radiation formula if one

1. adheres to equation (I), given above, between resonator energy and radiation pressure, which Planck derived from Maxwell's theory[2],
2. modifies the statistical theory of heat by the following assumption: A structure that is capable of oscillations with the frequency ν, and which, due to its possession of an electric charge, is capable of converting radiation energy into energy of matter and vice verse, cannot assume oscillation states of any arbitrary energy, but rather only such oscillation states whose energy is a multiple of $h\nu$. Where h is the constant designated by Planck which appears in his radiation equation.

6. Since the modification of the foundations of Planck's theory, just described, necessarily leads to very profound changes in our physical theories, it is very important to search for the simplest possible mutually independent interpretations of Planck's radiation formula as well as of the radiation law in general, insofar as the latter may be assumed to be known. Two considerations on this matter, which are distinguished by their simplicity, shall be briefly described below.

Until now, the equation $S = \frac{R}{N} \log W$ has been applied mainly to calculate the quantity W on the basis of a more or less complete theory, and then to calculate the entropy from W. However, this equation can also be applied conversely, using empirically obtained entropy values S_ν to obtain the statistical probability of the individual states A_ν of an isolated system. A theory yielding values for the probability of a state that differ from those obtained in this way must obviously be rejected. An analysis of the kind indicated for determining certain statistical properties of heat radiation enclosed in a cavity has already been carried out by me in an earlier paper[3], in which I first presented the theory of light quanta. However, since at that time I started from Wien's radiation formula, which is valid only in the limit (for large values of $\frac{\nu}{T}$), I shall present here a similar derivation which provides a simple interpretation of the meaning of Planck's radiation formula.

Let V and v be two interconnected spaces bounded by diffuse completely reflecting walls. Radiation energy within the frequency range $d\nu$ is enclosed in these spaces. H is the radiation energy existing instantaneously in V, and η the radiation energy existing instantaneously in v. After some time the proportion $h_0 : \eta_0 = V : v$ will, to within an approximation, be achieved. At an arbitrarily chosen instant of time, η will deviate from η_0 according to a statistical law that is obtained directly from the relation between S and W, then if one changes over to differentials

$$dW = \text{const} \times e^{\frac{N}{R} S} d\eta.$$

Let Σ and σ denote the entropy of the radiation in the two respective spaces and set $\eta = \eta_0 + \varepsilon$, then we have

$$d\eta = d\varepsilon$$

and

$$\begin{aligned} S &= \Sigma + \sigma \\ &= \Sigma_0 + \sigma_0 + \left\{ \frac{d(\Sigma + \sigma)}{d\varepsilon} \right\}_0 \varepsilon + \frac{1}{2} \left\{ \frac{d^2(\Sigma + \sigma)}{d\varepsilon^2} \right\}_0 \varepsilon^2 \dots \end{aligned}$$

The last equation goes to, since

$$\left\{ \frac{d(\Sigma + \sigma)}{d\varepsilon} \right\}_0 = 0$$

if one assumes that V is is very large compared with v,

$$S = \text{const} + \frac{1}{2} \left\{ \frac{d^2\sigma}{d\varepsilon^2} \right\}_0 \varepsilon^2 \dots$$

If we content ourselves with the first non vanishing term of the series, thus causing an error that is small, for larger v, compared with the cube of the radiation wavelength, we obtain

$$dW = \text{const} \times e^{-\frac{1}{2} \frac{N}{R} \left(\frac{d^2\sigma}{d\varepsilon^2} \right)_0 \varepsilon^2} d\varepsilon.$$

From this we obtain for the mean value $\overline{\varepsilon^2}$ of the square of the energy fluctuation of the radiation occurring in v,

$$\overline{\varepsilon^2} = \frac{1}{\frac{N}{R} \left\{ \frac{d^2\sigma}{d\varepsilon^2} \right\}_0}.$$

[1] A. Einstein, *Ann. d. Phys.* (4) **20,** 1906 and *Ann. d. Phys.* (4) **22,** 1907, §1.

[2] This amounts to the same as assuming that the electromagnetic theory of radiation at least yields correct time averages. This assumption can hardly be doubted in the light of the utility of this theory in optics.

[3] *Ann. d. Phys.* (4) **17,** 132-148, 1905.

If the radiation formula is known, we can calculate σ from it[1]. Taking Planck's radiation formula as an expression of experience, one obtains, after a simple calculation,

$$\overline{\varepsilon^2} = \frac{R}{Nk}\left\{\nu h\eta_0 + \frac{c^3}{8\pi\nu^2\,d\nu}\frac{\eta_0^2}{v}\right\}.$$

We have thus arrived at an easily interpretable expression for the mean value of the fluctuations of the radiation energy present in v. We shall now show that the current theory of radiation is incompatible with this result.

According to the current theory, the fluctuations are due solely to the circumstance that the infinitely many rays traversing the space, which constitute the radiation present in v, interfere with one another and thus provide a momentary energy that is sometimes greater, sometimes smaller than the sum of the energies that the individual rays would provide if they were not interfering with each other at all. We could thus exactly determine the quantity $\overline{\varepsilon^2}$ by a consideration that is mathematically somewhat complicated. We shall content ourselves here with a simple dimensional consideration. The following conditions must be satisfied:

1. The magnitude of the mean fluctuation depends only on λ (wavelength), $d\lambda$, σ and v, where σ denotes the radiation density associated with the wavelengths ($\sigma\,d\lambda = \varrho\,d\nu$).
2. Since the radiation energies of adjacent wavelength ranges and volumes[2] are simply additive, and the corresponding fluctuations are independent of each other, at a given λ and ϱ, $\overline{\varepsilon^2}$ must be proportional to the quantities $d\lambda$ and v.
3. $\overline{\varepsilon^2}$ has the dimension of energy squared.

The expression for $\overline{\varepsilon^2}$ is thereby completely determined up to a numerical factor (of order of magnitude 1). In this way one arrives at the expression $\sigma^2\lambda^4 v\,d\lambda$, which upon introduction of the variables used above reduces to the second term of the formula for $\overline{\varepsilon^2}$ just developed. But we would have obtained solely this second term for $\overline{\varepsilon^2}$ had we started out with the Jeans formula. One would then also have to set $\frac{R}{Nk}$ equal to a constant of order of magnitude 1, which corresponds to Planck's determination of the elementary quantum[3]. Thus, the first term of the above expression for $\overline{\varepsilon^2}$, which for the visible radiation surrounding us everywhere makes a far greater contribution than the second one, is not compatible with the current theory.

If one sets, following Planck, $\frac{R}{Nk} = 1$, then the first term, if present alone, would yield a fluctuation of the radiation energy equal to that produced if the radiation consisted of point quanta of energy $h\nu$ moving independently of each other. This can be shown by a simple calculation. One should remember that the contribution of the first term to the average percent fluctuation of energy

$$\left(\sqrt{\frac{\overline{\varepsilon^2}}{\eta_0^2}}\right)$$

is the greater the smaller the energy η_0, and that the magnitude of this percent fluctuation yielded by the first term is independent of the size of the space v over which the radiation is distributed; I mention this in order to show how fundamentally different the actual statistical properties of radiation are from those to be expected on the basis of our current theory, which is based on linear, homogeneous differential equations.

7. In the foregoing we have calculated the fluctuations of the energy distribution in order to obtain information on the nature of thermal radiation. In what follows we shall briefly show how one can obtain analogous results by calculating the fluctuations of the radiation *pressure,* due to fluctuations of the momentum.

Let a cavity surrounded on all sides by matter of absolute temperature T contain a mirror that can move freely in the direction perpendicular to its normal[4]. If we imagine it to be moving with a certain velocity from the outset, then, due to this motion, more radiation will be reflected at its front than at it's back; hence, the radiation pressure acting on the front will be greater than that acting on the back. Thus, due to its motion relative to the cavity radiation, the mirror will be acted upon by a force comparable to friction, which little by little would have to consume the momentum if there did not exist a cause of motion exactly compensating the average for the momentum lost through the above-mentioned frictional force. To the irregular fluctuations of the energy of a radiation space studied above, there also correspond irregular fluctuations of the momentum, or irregular fluctuations of the pressure forces exerted by the radiation

[1] Cf. Planck's often cited book, Equation (230).

[2] Naturally, only if these are large enough.

[3] By carrying out the interference consideration indicated above, one obtains $\frac{R}{Nk} = 1$.

[4] The motions of the mirror considered here are wholly analogous to the so-called Brownian motion of suspended particles.

on the mirror, which would have to set the mirror in motion even if it had originally been at rest. The mean speed of the mirror has then to be determined from the entropy-probability relation, and the law of the above-mentioned frictional forces from the radiation law, which is assumed to be known. From these two results one then calculates the effect of the pressure fluctuations, which in turn makes it possible to draw conclusions concerning the constitution of the radiation or—more precisely—concerning the elementary processes of the reflection of the radiation from the mirror.

Let ν denote the velocity of the mirror at time t. Owing to the frictional force mentioned above, this velocity decreases by $\frac{P\nu\tau}{m}$ in the small time interval τ, where m denotes the mass of the mirror and P the retarding force corresponding to unit velocity of the mirror. Further, we denote by Δ the velocity change of the mirror during τ corresponding to the irregular fluctuations of the radiation pressure. The velocity of the mirror at time $t+\tau$ is

$$v - \frac{P\tau}{m}v + \Delta.$$

For the condition that, on average, v shall remain unchanged during τ, we obtain

$$\overline{\left(v - \frac{P\tau}{m}v + \Delta\right)} = \overline{v^2}$$

or, if we omit relatively infinitesimal quantities and take into account that the average value of $v\Delta$ obviously vanishes:

$$\overline{\Delta^2} = \frac{2P\tau}{m}\overline{v^2}.$$

In this equation $\overline{v^2}$ can be replaced, using the equation

$$\frac{m\overline{v^2}}{2} = \frac{1}{2}\frac{RT}{N},$$

which can be derived from the entropy-probability equation. Before giving the value of the friction constant P, we specialize the problem under consideration by assuming that the mirror completely reflects the radiation of a certain frequency range (between ν and $\nu + d\nu$) and is completely transparent to radiation of other frequencies. By calculation omitted here for the sake of brevity, one obtains from a purely electrodynamic investigation the following equation, which is valid for any arbitrary radiation distribution:

$$P = \frac{3}{2c}\left[\varrho - \frac{1}{3}\nu\frac{d\varrho}{d\nu}\right]d\nu f,$$

where ϱ again denotes the radiation density at frequency ν, and f denotes the surface area of the mirror. By substituting the values obtained for $\overline{v^2}$ and P, we get

$$\frac{\overline{\Delta^2}}{\tau} = \frac{RT}{N}\frac{3}{c}\left[\varrho - \frac{1}{3}\nu\frac{d\varrho}{d\nu}\right]d\nu f.$$

If we transform this expression using Planck's radiation formula, we obtain

$$\frac{\overline{\Delta^2}}{\tau} = \frac{1}{c}\left[h\varrho\nu + \frac{c^3}{8\pi}\frac{\varrho^2}{\nu^2}\right]d\nu f.$$

The close connection between this relation and the one derived in the last section for the energy fluctuation ($\overline{\varepsilon^2}$) is immediately obvious[1], and exactly analogous considerations can be applied to it. Further, according to the current theory, the expression must reduce to the second term (fluctuation due to interference). If the first term alone were present, the fluctuations of the radiation pressure could be completely explained by the assumption that the radiation consists of independently moving, not too extended, complexes of energy $h\nu$. In this case, too, the formula says that in accordance with Planck's formula the effects of the two mentioned causes of fluctuation act like fluctuations (errors) arising from mutually independent causes (additivity of the terms of which the square of the fluctuation is composed).

8. In my opinion, the last two considerations conclusively show that the constitution of radiation must be different from what we currently believe. It is true that, as the excellent agreement of theory and experiment in optics has proved, our current theory correctly yields the time averages, which alone can be directly observed, but it necessarily leads to laws on thermal properties of radiation that prove to be incompatible with experience if one maintains the entropy-probability relation. The discrepancy between the phenomena and the theory is the

[1] These relations can be written in the form (with $\frac{R}{Nk} = 1$)

$$\overline{\varepsilon^2} = \left\{h\varrho\nu + \frac{c^3\varrho^2}{8\pi\nu^2}\right\}v\,d\nu.$$

more prominent the larger ν and the smaller ϱ. At small ϱ the temporal fluctuations of the radiation energy of a given space or of the force of radiation pressure on a given surface are much larger than expected from our current theory.

We have seen that Planck's radiation law can be understood if one uses the assumption that the oscillation energy of frequency ν can occur only in quanta of magnitude $h\nu$. According to the aforesaid, it is not sufficient to assume that radiation can only be *emitted and absorbed* in quanta of this magnitude, i.e., that we are dealing with a property of the emitting or absorbing matter only; considerations **6.** and **7.** show that the fluctuations in the spatial distribution of the radiation and in the radiation pressure also occur as if the radiation consisted of quanta of the indicated magnitude. Certainly, it cannot be asserted that the quantum theory follows from Planck's radiation law as a consequence and that other interpretations are excluded. However, one can well assert that the quantum theory provides the simplest interpretation of the Planck formula.

It should be emphasized that the considerations presented would in the main in no way lose their value if it should turn out that Planck's formula is not valid; it is precisely that part of Planck's formula which has been adequately confirmed by experience (the Wien radiation law valid in the limit for large $\frac{\nu}{T}$) which leads to the theory of the light quantum.

9. The experimental investigation of the consequences of the theory of light quanta is, in my opinion, one of the most important tasks that the experimental physics of today must solve. The results obtained so far can be divided into three groups

a) There are clues concerning the energy of those elementary processes that are associated with the absorption or emission of radiation of a certain frequency (Stokes' rule; velocity of cathode rays produced by light or X-rays; cathode luminescence, etc). To this group also belongs the interesting use Mr. Stark has made of the theory of light quanta to elucidate the peculiar energy distribution in the spectrum of a spectral line emitted by channel rays[1].

The method of deduction is always as follows: If one elementary process produces another one, then the energy, of the latter is not larger than that of the former. On the other hand, the energy of one of the two elementary processes is known (of magnitude $h\nu$) if the latter consists in the absorption or emission of radiation of a specified frequency.

Especially interesting would be the study of exceptions to Stokes' law. In order to explain these exceptions, one has to assume that a light quantum is emitted only when the emission center in question has absorbed two light quanta. The frequency of such an event, and thus also the intensity of the emitted light having a smaller wavelength than the producing one, will in this case have to be proportional to the square of the intensity of the exciting light at weak irradiation (according to the law of mass action), while according to Stokes' rule a proportionality with the first power of the exciting light intensity is to be expected at weak irradiation.

b) If the absorption[2] of each light quantum brings about an elementary process of a certain kind, then $\frac{E}{h\nu}$ is the number of these elementary processes if the quantity of energy E of radiation of frequency ν is absorbed.

Thus, for example, if the quantity E of radiation of frequency ν is absorbed by a gas being ionized, then it is to be expected that $\frac{E}{Nh\nu}$ gram molecules of the gas will be ionized. This relation only appears to presume the knowledge of N; for if Planck's radiation formula is written in the form

$$\varrho = \alpha\nu^3 \frac{1}{e^{\frac{\beta\nu}{T}} - 1},$$

then $\frac{E}{R\beta\nu}$ is the number of ionized gram molecules.

This relation, which I presented in my first paper[3] on this subject, has unfortunately remained unnoticed thus far.

c) The results noted in **5.** lead to a modification of the kinetic theory of specific heat[4] and to certain relations between the optical and the thermal behavior of bodies.

10. It seems difficult to set up a theoretical system that interprets the light quanta in a complete fashion, the way our current molecular mechanics in conjunction with the Maxwell-Lorentz theory is able to interpret the radiation formula propounded by Mr. Jeans. That we

[1] J. Stark, *Phys. Zeit.* **9,** 767, 1908.

[2] Of course, the analogous consideration holds also conversely for the production of light by elementary processes (e.g., by collisions of ions).

[3] *Ann. d. Phys.* (4) **17,** 132-148, 1905, §9. ed: In this paper, β is defined as $\beta = \frac{h}{k}$.

[4] A. Einstein, *Ann. d. Phys.* (4) **22,** 1907, pgs. 180-190 and 800.

are only dealing with a *modification* of our current theory, not with its complete *abolition,* seems already to be implied by the fact that Jeans' law seems to be valid in the limit (for small $\frac{\nu}{T}$). An indication as to how this modification would have to be carried out is given by a dimensional analysis carried out by Mr. Jeans a few years ago, which is extremely important, in my opinion, and which—modified in some points—I shall now recount in brief.

Imagine that a closed space contains an ideal gas and radiation and ions, and that owing to their charge, the ions are able to mediate an energy exchange between gas and radiation. In a theory of radiation linked with the consideration of this system the following quantities can be expected to play a role, i.e., to appear in the expression to be obtained for the radiation density ϱ :

a) the mean energy η of a molecular structure (up to an unknown numerical factor like $\frac{RT}{N}$),

b) the velocity of light c,

c) the elementary quantum ε of electricity,

d) the frequency ν.

From the dimension of ϱ, by solely considering the dimensions of the four quantities mentioned above, one can determine in a simple way what the form of the expression for ϱ must be. Substituting the value of $\frac{RT}{N}$ for η, we obtain

$$\varrho = \frac{\varepsilon^2}{c^4}\nu^3\psi(\alpha),$$

where

$$\alpha = \frac{R\varepsilon^2}{Nc}\frac{\nu}{T},$$

and where ψ denotes a function that remains undetermined. This equation contains the Wien displacement law, whose validity can hardly remain in doubt. This has to be understood as a confirmation of the fact that apart from the four quantities introduced above, no other dimensional quantities play a role in the radiation law.

From this we conclude that, except for dimensionless numerical factors that appear in theoretical developments and of course cannot be determined by dimensional considerations, the coefficients $\frac{\varepsilon^2}{c^4}$ and $\frac{R\varepsilon^2}{Nc}$ appearing in the equation for ϱ must be numerically equal to the coefficients appearing in the Planck (or Wien) radiation formula. Since the above non determinable dimensionless numerical factors are hardly likely to make a change in the order of magnitude, we can set, within the order of magnitude[1] is concerned

$$\frac{h}{c^3} = \frac{\varepsilon^2}{c^4} \qquad \text{and} \qquad \frac{h}{k} = \frac{R}{N}\frac{\varepsilon^2}{c},$$

hence

$$h = \frac{\varepsilon^2}{c} \qquad \text{and} \qquad k = \frac{N}{R}.$$

It is the second of these equations which has been used by Mr. Planck to determine the elementary quanta of matter or electricity. Concerning the expression for h, it should be noted that

$$h = 6 \times 10^{-27}$$

but

$$\frac{\varepsilon^2}{c} = 7 \times 10^{-30}.$$

There are three decimal places missing here. But, this may be due to the fact that the dimensionless factors are not known.

The most important aspect of this derivation is that it relates the light quantum constant h to the elementary quantum ε of electricity. We should remember that the elementary quantum ε is a stranger in Maxwell-Lorentz electrodynamics[2]. Foreign forces must be enlisted in order to construct the electron in the theory; usually, one introduces a rigid framework to prevent the electron's electrical masses from flying apart under the influence of their electric interaction. The relation $h = \frac{\varepsilon^2}{c}$ seems to me to indicate that the same modification of the theory that will contain the elementary quantum ε as a consequence will also contain the quantum structure of radiation as a consequence. The fundamental equation of optics

$$D(\varphi) = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial^2 t} - \left[\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right] = 0$$

will have to be replaced by an equation in which the universal constant ε (probably its square) also appears in a coefficient. The equation sought (or the system of equations sought) must be homogeneous in its dimensions. It must remain unchanged upon application of the Lorentz transformation. It cannot be linear and homogeneous. It must—at least if Jeans' law is really valid in the limit

[1] The Planck formula reads: $\varrho = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{\frac{h\nu}{kT}}-1}$.

[2] Cf. Levi-Civita, *Comptes Rendus,* 1907, "Sur le mouvement etc."

for small $\frac{\nu}{T}$—lead in the limit for large amplitudes to the form $D(\varphi) = 0$.

I have not yet succeeded in finding a system of equations fulfilling these conditions which would have looked to me suitable for the construction of the elementary electrical quantum and the light quanta. The variety of possibilities does not seem so great, however, for one to be forced to shrink from this task.

Addendum

From what has been said in this paper under **4.** above, the reader could easily get an incorrect impression about the standpoint taken by Mr. Planck with regard to his own theory of thermal radiation. I therefore deem it appropriate to note the following.

In his book, Mr. Planck emphasized in several places that his theory should not yet be viewed as something complete and final. At the end of his introduction, for example, he says verbatim: "I find it important, however, to especially emphasize at this point the fact, as elaborated in greater detail in the last paragraphs of the book, that the theory developed here does not claim by any means to be fully complete, even though, as I believe, it offers a feasible approach by which to consider the processes of energy radiation from the same viewpoint as those of molecular motion."

The pertinent discussions in my paper should not be construed as an objection (in the strict sense of the word) against Planck's theory, but rather as an attempt to formulate and apply the entropy-probability principle more rigorously than has been done untill now. A more rigorous formulation of this principle was necessary because without it the subsequent developments in the paper, in which the molecular structure of radiation was deduced, would not have been adequately substantiated. So that my conception of the principle would not appear as chosen somewhat ad hoc, or arbitrary, I had to show why the existing formulation is not yet completely satisfying.

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