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ON THE RELATION BETWEEN THE EXPANSION AND THE MEAN DENSITY OF THE UNIVERSE

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In a recent note in the *Göttinger Nachrichten*, Dr. O. Heckmann has pointed out that the non-static solutions of the field equations of the general theory of relativity with constant density do not necessarily imply a positive curvature of three-dimensional space, but that this curvature may also be negative or zero.

There is no direct observational evidence for the curvature, the only directly observed data being the mean density and the expansion, which latter proves that the actual universe corresponds to the non-statical case. It is therefore clear that from the direct data of observation we can derive neither the sign nor the value of the curvature, and the question arises whether it is possible to represent the observed facts without introducing a curvature at all.

Historically the term containing the "cosmological constant" λ was introduced into the field equations in order to enable us to account theoretically for the existence of a finite mean density in a static universe. It now appears that in the dynamical case this end can be reached without the introduction of λ.

If we suppose the curvature to be zero, the line-element is

$$ds^2 = -R^2(dx^2 + dy^2 + dz^2) + c^2dt^2, \tag{1}$$

where R is a function of t only, and c is the velocity of light. If, for the sake of simplicity, we neglect the pressure p,[1] the field equations without λ lead to two differential equations, of which we need only one, which in the case of zero curvature reduces to:

$$\frac{1}{R^2}\left(\frac{dR}{cdt}\right)^2 = \frac{1}{3}\kappa\rho. \tag{2}$$

The observations give the coefficient of expansion and the mean density:

$$\frac{1}{R}\frac{dR}{cdt} = h = \frac{1}{R_B}; \quad \rho = \frac{2}{\kappa R_A^2}.$$

Therefore we have, from (2), the theoretical relation

$$h^2 = \frac{1}{3}\kappa\rho \tag{3}$$

or

$$\frac{R_A^2}{R_B^2} = \frac{2}{3}. \tag{3'}$$

Taking for the coefficient of expansion

$$h = 500 \text{ km./sec. per } 10^6 \text{ parsecs,} \tag{4}$$

or

$$R_B = 2 \times 10^{27} \text{ cm.,}$$

we find

$$R_A = 1.63 \times 10^{27} \text{ cm.,}$$

or

$$\rho = 4 \times 10^{-28} \text{ gr. cm.}^{-3}, \tag{5}$$

which happens to coincide exactly with the upper limit for the density adopted by one of us.[2]

The determination of the coefficient of expansion h depends on the measured red-shifts, which do not introduce any appreciable uncertainty, and the distances of the extra-galactic nebulae, which are still very uncertain. The density depends on the assumed masses of these nebulae and on the scale of distance, and involves, moreover, the assumption that all the material mass in the universe is concentrated in the nebulae. It does not seem probable that this latter assumption will introduce any appreciable factor of uncertainty. Admitting it, the ratio h^2/ρ, or R_A^2/R_B^2 as derived from observations, becomes proportional to Δ/M, Δ being the side of a cube containing on the average one nebula, and M the average mass of the nebulae. The values adopted above would correspond to $\Delta = 10^6$ light years, $M = 2.10^{11}\odot$, which is about Dr. Oort's estimate of the mass of our own galactic system. Although, therefore, the density (5) corresponding to the assumption of zero curvature and to the coefficient of expansion (4) may perhaps be on the high side, it certainly is of the correct order of magnitude, and we must conclude that at the present time it is possible to represent the facts without assuming a curvature of three-dimensional space. The curvature is, however, essentially determinable, and an increase in the precision of the data derived from observations will enable us in the future to fix its sign and to determine its value.

[1] It seems certain that the pressure p in the actual universe is negligible as compared with the material density ρ_0. The same reasoning, however, holds good if the pressure is not neglected.

[2] *Bull. Astronom. Inst. Netherlands, Haarlem,* **6,** 142 (1931).