DAVID HILBERT

THE FOUNDATIONS OF PHYSICS (SECOND COMMUNICATION)

Originally published as "Die Grundlagen der Physik. (Zweite Mitteilung)" in Nachrichten von der Königlichen Gesellschaft zu Göttingen. Math.-phys. Klasse. 1917, p. 53–76. Presented in the Session of 23 December 1916.

In my first communication[1] I proposed a system of basic equations of physics. Before turning to the theory of integrating these equations it seems necessary to discuss some more general questions of a logical as well as physical nature.

First we introduce in place of the world parameters w_s $(s = 1, 2, 3, 4)$ the most general *real* spacetime coordinates x_s $(s = 1, 2, 3, 4)$ by putting

$$w_1 = x_1, \qquad w_2 = x_2, \qquad w_3 = x_3, \qquad w_4 = x_4,$$

and correspondingly in place of

$$ig_{14}, \qquad ig_{24}, \qquad ig_{34}, \qquad -g_{44},$$

we write simply

$$g_{14}, \qquad g_{24}, \qquad g_{34}, \qquad g_{44}.$$

The new $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ —the gravitational potentials of Einstein—shall then all be real functions of the real variables x_s $(s = 1, 2, 3, 4)$ of such a type that, in the representation of the quadratic form

$$G(X_1, X_2, X_3, X_4) = \sum_{\mu\nu} g_{\mu\nu} X_\mu X_\nu \tag{28}$$

as a sum of four squares of linear forms of the X_s, three squares always occur with positive sign, and one square with negative | sign: thus the quadratic form (28) provides our four dimensional world of the x_s with the metric of a pseudo-geometry. [54]
The determinant g of the $g_{\mu\nu}$ turns out to be negative.

1 This journal, 20 November 1915.

Jürgen Renn (ed.). *The Genesis of General Relativity,* Vol. 4
Gravitation in the Twilight of Classical Physics: The Promise of Mathematics.

above; rather, according to Theorem I, four of them are a consequence of the rest: we regarded the four Maxwell equations (5) as a consequence of the ten gravitational equations (4), and so we have for the 14 potentials $g_{\mu\nu}$, q_s only 10 equations (4) that are essentially independent of each other. |

As soon as we maintain the demand of general invariance for the basic equations [60]
of physics the circumstance just mentioned is essential and even necessary. Because if there were further invariant equations, independent of (4), for the 14 potentials, then introduction of a Gaussian coordinate system would lead for the 10 physical quantities as per (33),

$$g_{\mu\nu} \quad (\mu, \nu = 1, 2, 3), \qquad q_s \quad (s = 1, 2, 3, 4)$$

to a system of equations that would again be mutually independent, and mutually contradictory, because there are more than 10 of them.

Under such circumstances then, as occur in the new physics of general relativity, it is by no means any longer possible from knowledge of physical quantities in present and past to derive uniquely their future values. To show this intuitively on an example, let our basic equations (4) and (5) of the first communication be integrated in the special case corresponding to the presence of a single electron permanently at rest, so that the 14 potentials

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}(x_1, x_2, x_3) \\ q_s &= q_s(x_1, x_2, x_3) \end{aligned}$$

become definite functions of x_1, x_2, x_3, all independent of the time x_4, and in addition such that the first three components r_1, r_2, r_3 of the four-current density vanish. Then we apply the following coordinate transformation to these potentials:

$$\begin{cases} x_1 = x'_1 & \text{for } x'_4 \leq 0 \\ x_1 = x'_1 + e^{-\frac{1}{x'^2_4}} & \text{for } x'_4 > 0 \end{cases}$$

$$\begin{aligned} x_2 &= x'_2 \\ x_3 &= x'_3 \\ x_4 &= x'_4. \end{aligned}$$

For $x'_4 \leq 0$ the transformed potentials $g'_{\mu\nu}$, q'_s are the same functions of x'_1, x'_2, x'_3 as the $g_{\mu\nu}$, q_s of the original variables x_1, x_2, x_3, whereas the $g'_{\mu\nu}$, q'_s for $x'_4 > 0$ depend in an essential way also on the time coordinate x'_4; that is, the potentials $g'_{\mu\nu}$, q'_s represent an electron that is at rest until $x'_4 = 0$, but then puts its components into motion. |

Nonetheless I believe that it is only necessary to formulate more sharply the idea [61]
on which the principle of general relativity[3] is based, in order to maintain the principle of causality also in the new physics. Namely, to follow the essence of the new rel-

ativity principle we must demand invariance not only for the general laws of physics, but we must accord invariance to each separate statement in physics that is to have physical meaning—in accordance with this, that in the final analysis it must be possible to establish each physical fact by thread or light clock, that is, instruments of *invariant* character. In the theory of curves and surfaces, where a statement in a chosen parametrization of the curve or surface has no geometrical meaning for the curve or surface itself, if this statement does not remain invariant under any arbitrary transformation of the parameters or cannot be brought to invariant form; so also in physics we must characterize a statement that does not remain invariant under any arbitrary transformation of the coordinate system as *physically meaningless*. For example, in the case considered above of the electron at rest, the statement that, say at the time $x_4 = 1$ this electron is at rest, has no physical meaning because this statement is not invariant.

Concerning the principle of causality, let the physical quantities and their time derivatives be known at the present in some given coordinate system: then a statement will only have physical meaning if it is invariant under all those transformations, for which the coordinates just used for the present remain unchanged; I maintain that statements of this type for the future are all uniquely determined, that is, *the principle of causality holds in this form*:

From present knowledge of the 14 physical potentials $g_{\mu\nu}$, q_s all statements about them for the future follow necessarily and uniquely provided they are physically meaningful.

To prove this proposition we use the *Gaussian* spacetime coordinate system. Introducing (33) into the basic equations (4) of the first communication yields for the 10 potentials |

[62] $$g_{\mu\nu} \quad (\mu, \nu = 1, 2, 3), \qquad q_s \quad (s = 1, 2, 3, 4) \tag{34}$$

a system of as many partial differential equations; if we integrate these on the basis of the given initial values at $x_4 = 0$, we find uniquely the values of (34) for $x_4 > 0$. Since the Gaussian coordinate system itself is uniquely determined, therefore also all statements about those potentials (34) with respect to these coordinates are of invariant character.

The forms, in which physically meaningful, i.e. invariant, statements can be expressed mathematically are of great variety.

First. This can be done by means of an invariant coordinate system. Like the Gaussian system used above one can apply the well-known Riemannian one, as well as that spacetime coordinate system in which electricity appears at rest with unit current density. As at the end of the first communication, let $f(q)$ denote the function occurring in Hamilton's principle and depending on the invariant

3 In his original theory, now abandoned, A. Einstein (*Sitzungsberichte der Akad. zu Berlin*, 1914, p. 1067) had indeed postulated certain 4 non-invariant equations for the $g_{\mu\nu}$, in order to save the causality principle in its old form.

$$q = \sum_{kl} q_k q_l g^{kl},$$

then

$$r^s = \frac{\partial f(q)}{\partial q_s}$$

is the four-current density of electricity; it represents a contravariant vector and therefore can certainly be transformed to $(0, 0, 0, 1)$, as is easily seen. If this is done, then from the four equations

$$\frac{\partial f(q)}{\partial q_s} = 0 \quad (s = 1, 2, 3), \qquad \frac{\partial f(q)}{\partial q_4} = 1$$

the four components of the four-potential q_s can be expressed in terms of the $g_{\mu\nu}$, and every relation between the $g_{\mu\nu}$ in this or in one of the first two coordinate systems is then an invariant statement. For particular solutions of the basic equations there may be special invariant coordinate systems; for example, in the case treated below of the centrally symmetric gravitational field r, ϑ, φ, t form an invariant system of coordinates up to rotations.

Second. The statement, *according to which a coordinate system can be found* in which the 14 potentials $g_{\mu\nu}$, q_s have certain definite values in the future, or fulfill certain definite conditions, is always an invariant and therefore a physically meaningful one. The mathematically invariant expression for | such a statement is obtained by [63] eliminating the coordinates from those relations. The case considered above, of the electron at rest, provides an example: the essential and physically meaningful content of the causality principle is here expressed by the statement that the electron which is at rest for the time $x_4 \leq 0$ will, *for a suitably chosen spacetime coordinate system*, also remain at rest in all its parts for the future $x_4 > 0$.

Third. A statement is also invariant and thus has physical meaning if it is supposed to be valid in any arbitrary coordinate system. An example of this are Einstein's energy-momentum equations having divergence character. For, although Einstein's energy does not have the property of invariance, and the differential equations he put down for its components are by no means covariant as a system of equations, nevertheless the assertion contained in them, that they shall be satisfied in any coordinate system, is an invariant demand and therefore it carries physical meaning.

According to my exposition, physics is a four-dimensional pseudo-geometry, whose metric $g_{\mu\nu}$ is connected to the electromagnetic quantities, i.e. to the matter, by the basic equations (4) and (5) of my first communication. With this understanding, an old geometrical question becomes ripe for solution, namely whether and in what sense Euclidean geometry—about which we know from mathematics only that it is a logical structure free from contradictions—also possesses validity in the real world.

The old physics with the concept of absolute time took over the theorems of Euclidean geometry and without question put them at the basis of every physical theory. Gauss as well proceeded hardly differently: he constructed a hypothetical non-

Euclidean physics, by maintaining the absolute time and revoking only the parallel axiom from the propositions of Euclidean geometry; a measurement of the angles of a triangle of large dimensions showed him the invalidity of this non-Euclidean physics.

The new physics of Einstein's principle of general relativity takes a totally different position vis-à-vis geometry. It takes neither Euclid's nor any other particular geometry *a priori* as basic, in order to deduce from it the proper laws of physics, but,
[64] as I showed in my first communication, | the new physics provides at one fell swoop through one and the same Hamilton's principle the geometrical and the physical laws, namely the basic equations (4) and (5), which tell us how the metric $g_{\mu\nu}$ —at the same time the mathematical expression of the phenomenon of gravitation—is connected with the values q_s of the electrodynamic potentials.

Euclidean geometry is *an action-at-a-distance law foreign to the modern physics*: By revoking the Euclidean geometry as a general presupposition of physics, the theory of relativity maintains instead that geometry and physics have identical character and are based as *one* science on a common foundation.

The geometrical question mentioned above amounts to the investigation, whether and under what conditions the four-dimensional Euclidean pseudo-geometry

$$\begin{aligned} &g_{11} = 1, \qquad g_{22} = 1, \qquad g_{33} = 1, \qquad g_{44} = -1 \\ &g_{\mu\nu} = 0 \qquad (\mu \neq \nu) \end{aligned} \tag{35}$$

is a solution, or even the only regular solution, of the basic physical equations.

The basic equations (4) of my first communication are, due to the assumption (20) made there:

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0,$$

where

$$[\sqrt{g}K]_{\mu\nu} = \sqrt{g}\Big(K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu}\Big).$$

When the values (35) are substituted, we have

$$[\sqrt{g}K]_{\mu\nu} = 0 \tag{36}$$

and for

$$q_s = 0 \qquad (s = 1, 2, 3, 4)$$

we have

$$\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0;$$

that is, when all electricity is removed, the pseudo-Euclidean geometry is possible. The question whether it is also necessary in this case, i.e. whether—or under certain