

Albert Einstein

"On the Inertia  
of Energy

Required by the  
Relativity  
Principle"

Annalen der Physik

23 (1907), pp. 371-384

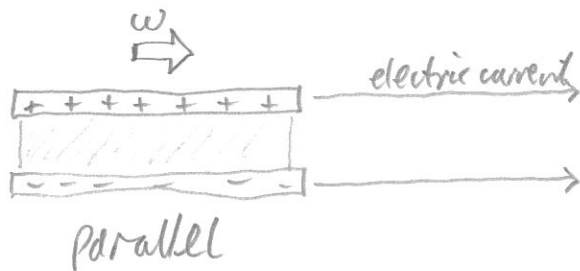
Notes by John D. Norton  
September 2015

# Introductory puzzle NOT mentioned by Einstein

Trouton & Noble  
1903

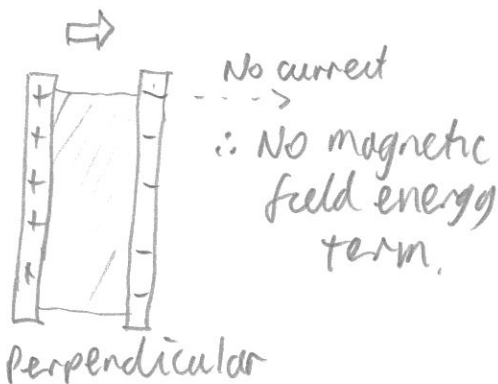
The other second order ether drift experiment

charged capacitor moves at  $w$  in ether



magnetic field created with energy

$$\sqrt{\frac{w}{c}}^2 \text{ Electro-static energy of capacitor}$$



Energy parallel

>

Energy perpendicular

Expect capacitor to turn to this orientation

capacitor suspended by wire

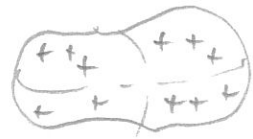


no turning observed

How do we new explain this?

# Main Results

## 1. Kinetic energy of stressed body



$$K = \underbrace{\left[ \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 \right]}_{\text{familiar term}} +$$

term due to stress in the body

$$\Delta E = \frac{v^2}{c^2} \frac{1}{\sqrt{1-v^2/c^2}} (\text{Rest volume}) (\text{Normal stress in direction motion})$$

Energy NOT due to elastic deformation

2. Rigid bodies require signals to propagate at speeds > speed light. Possibility of signals arriving before they leave "impossible"

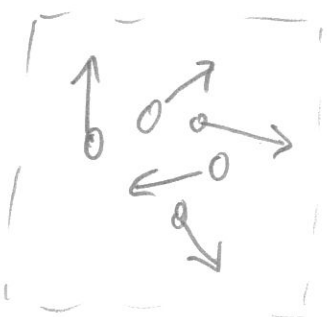
[ (novel?) energy argument ]

3. Collection of moving masses:

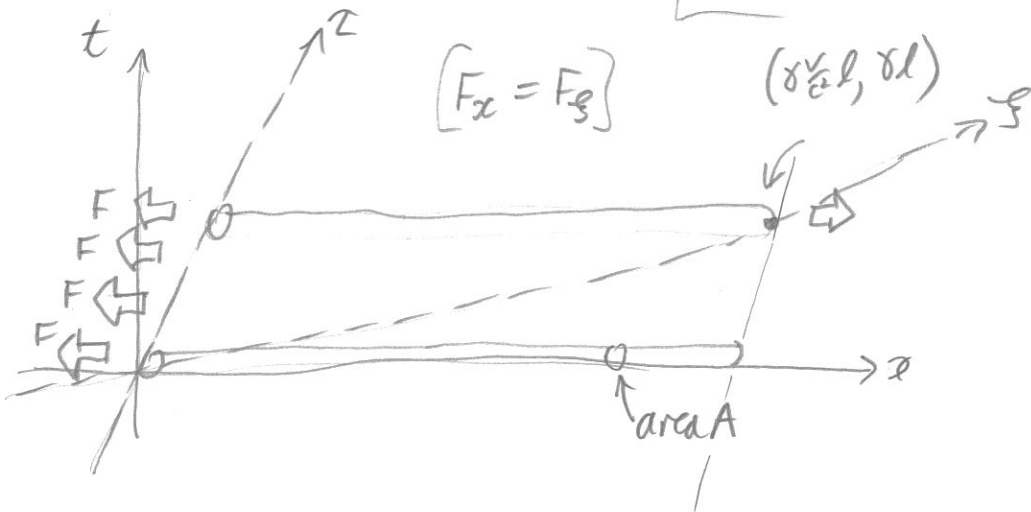
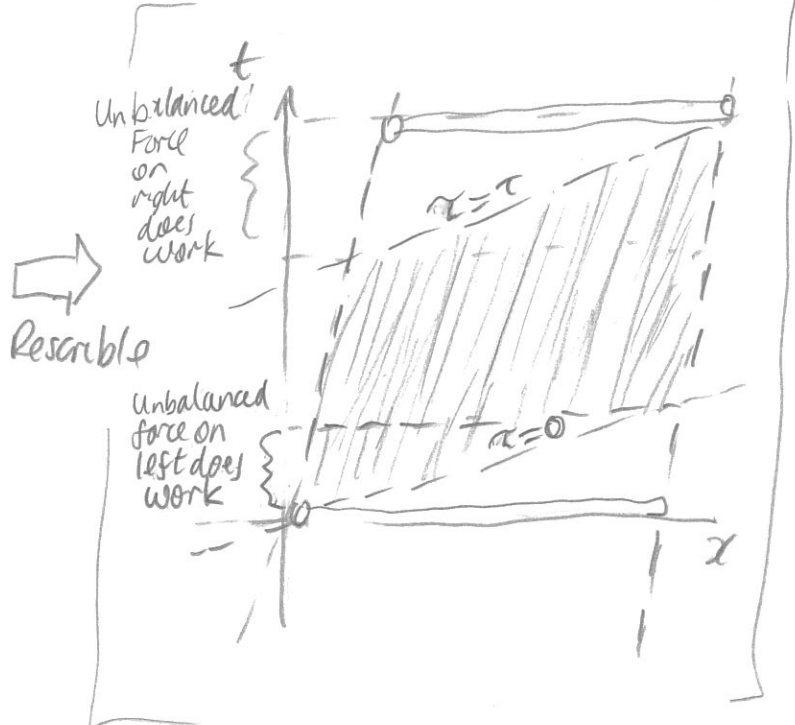
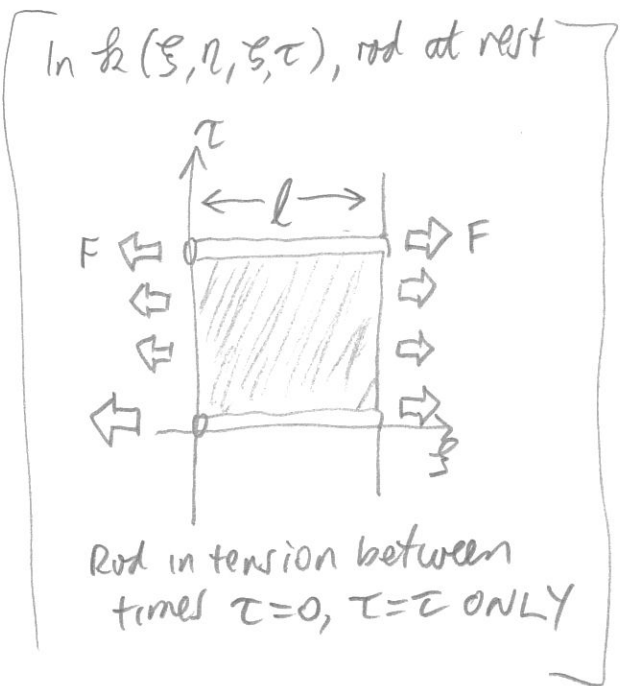
Kinetic energy of each mass

contributes to

Rest mass of the total system



# Simplified version of result 1



$$\begin{aligned} \tau=0 \\ \xi=l \\ \downarrow \\ t = \gamma(\tau + v/c^2 \xi) \\ = \gamma v/c^2 l \\ x = \gamma(\xi + v\tau) \\ = \gamma l \\ \gamma = 1/\sqrt{1-v^2/c^2} \end{aligned}$$

(work done on rod during time  $\gamma v/c^2 l$  of unbalanced force)

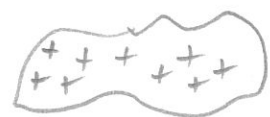
$$= -F \cdot v \cdot \gamma v/c^2 l$$

$$= -\frac{F}{A} \cdot A l \cdot \frac{v^2}{c^2} \cdot \gamma$$

$$= (\text{Normal stress}) (\text{Rest volume}) \frac{v^2/c^2}{\sqrt{1-v^2/c^2}} = \Delta E \text{ energy change in rod}$$

# Einstein's more general case

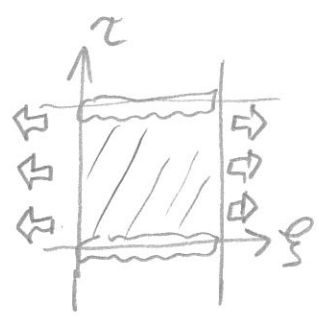
$K(\xi, \eta, \zeta)$  Rest frame of electrically charged body



charge density  $\rho'$   
Net force in x-direction

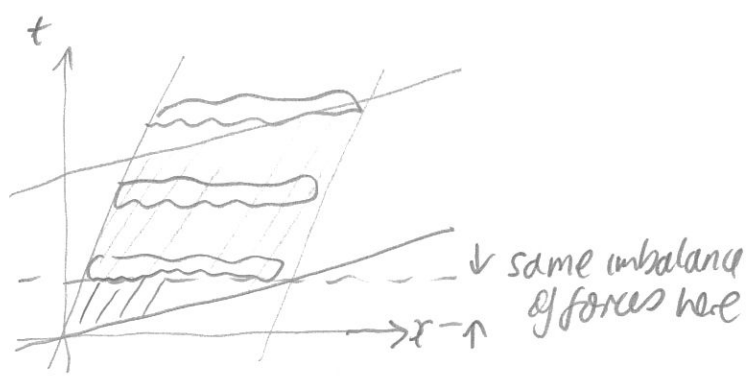
$$= \int x' \frac{\rho'}{4\pi} d\xi d\eta d\zeta$$

= 0 Equilibrium  $x' =$  x-component of  $\underline{E}$



External field present only in  $z=0$  to  $z=Z$

Transform to  $K(x, y, z, t)$  where body moves at  $v$  in x-direction



compute energy during brief period of unbalanced force:

(Long, messy integration)

$$\Delta E = - \frac{(v/c)^2}{\sqrt{1-v^2/c^2}} \sum \xi K_x$$

$\xi$  component of force on rest volume element  $d\xi d\eta d\zeta$   
 $= \frac{x' \rho'}{4\pi} \cdot d\xi d\eta d\zeta$

proper position in body

sum over all volume elements

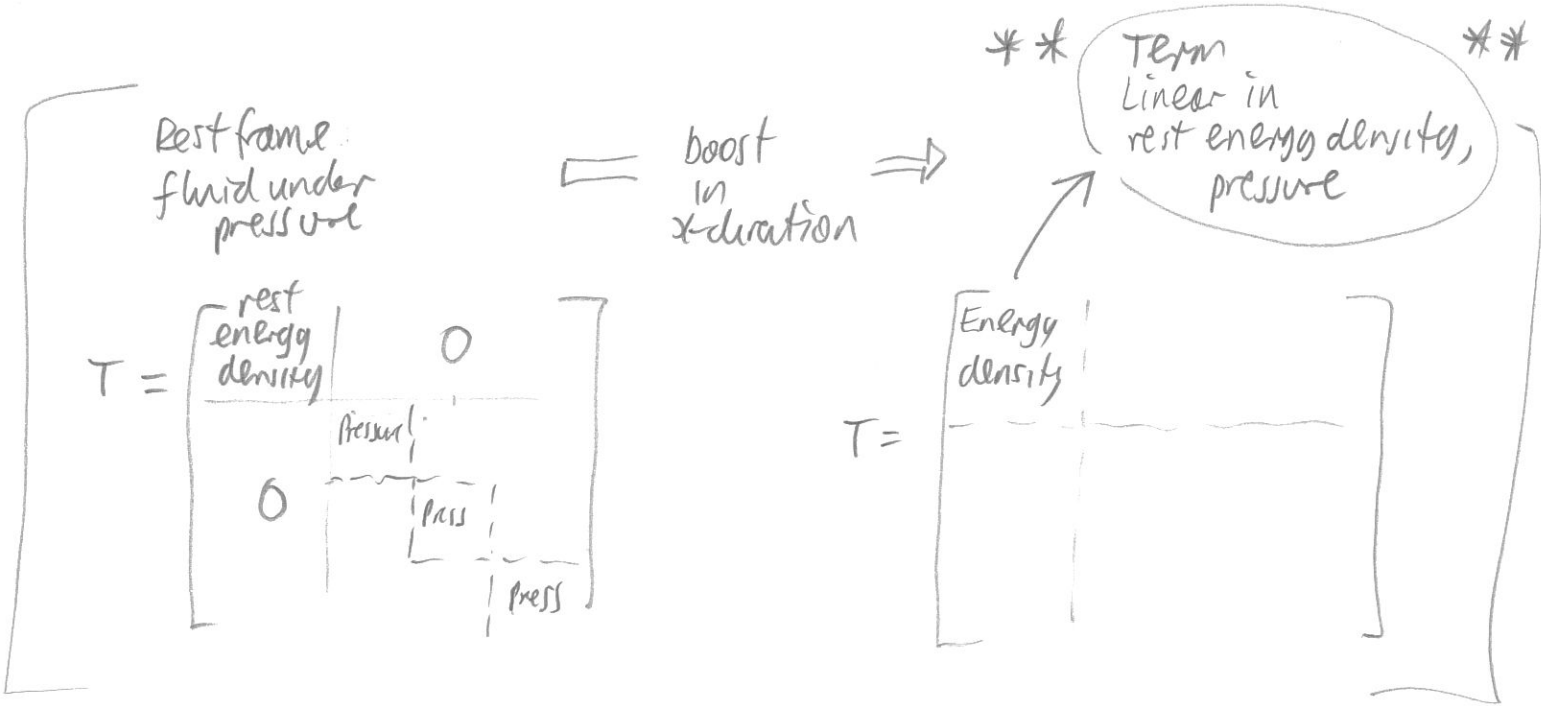
# Long Term Significance

Energy density represented as 0-0 component (STRESS-energy tensor)

Lorentz transformation mixes energy density & stresses

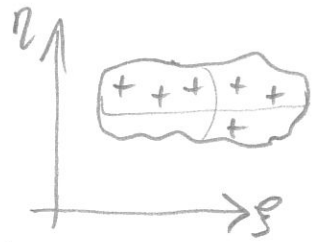
T =

Energy density	Energy flux = momentum density	
Energy flux	Normal stress	shear stress
momentum density	shear stress	Normal stress
		Normal stress



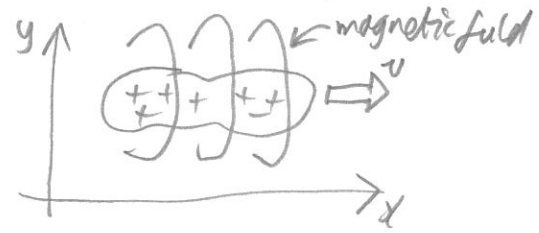
# Apparent Anisotropy of e-m self energy of charged body

Body at rest in  $k(\xi, \eta, \xi, z)$



Electric field  $(X', Y', Z') \neq 0$   
 magnetic  $(L', M', N') = 0$   
 Electrostatic system

Body moves at  $v$  in +x direction of  $K(x, y, z, t)$



From 1905 special relativity paper  
 $X = X'$                        $L = L'$   
 $Y = \beta(Y' + \frac{v}{c}N')$        $M = \beta(M' - \frac{v}{c}Z')$   
 $Z = \beta(Z' - \frac{v}{c}M')$        $N = \beta(N' + \frac{v}{c}Y')$

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Total electromagnetic self-energy in  $K$  =  $E_e = \frac{1}{8\pi} \int (X^2 + Y^2 + Z^2 + L^2 + M^2 + N^2) dx dy dz$



Substitute  $k$  frame values in formula

$$dx = \sqrt{1 - \frac{v^2}{c^2}} ds = \frac{1}{\beta} ds \quad X = X' \quad L = L'$$

$$\left. \begin{aligned} Y &= \beta(Y' + \frac{v}{c}N') = \beta Y' \\ N &= \beta(N' + \frac{v}{c}Y') = \beta \frac{v}{c} Y' \end{aligned} \right\} \begin{aligned} Y^2 + N^2 &= \beta^2 (Y'^2 + (\frac{v}{c}Y')^2) \\ &= \frac{1 + (\frac{v}{c})^2}{1 - (\frac{v}{c})^2} Y'^2 \end{aligned}$$

similar

$$Z^2 + M^2 = \frac{1 + (\frac{v}{c})^2}{1 - (\frac{v}{c})^2} Z'^2$$

$$E_e = \frac{1}{8\pi} \int \frac{1}{\beta} \left[ X'^2 + \frac{1 + (\frac{v}{c})^2}{1 - (\frac{v}{c})^2} (Y'^2 + Z'^2) \right] ds d\eta ds$$

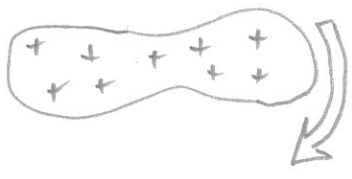
Self-energy of moving charged body varies with orientation w.r.t. to motion

} JON (Einstein does not say this)  
 Effect sought in Trouton-Noble experiment

# Einstein's Worry

# contradiction with principle of relativity

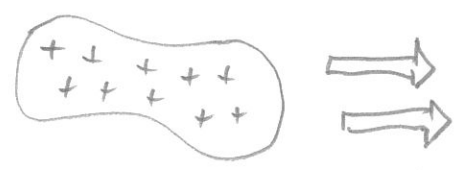
Rest frame



charged body can rotate "infinitely slowly"



Frame in which body moves uniformly



charged body cannot rotate infinitely slowly but will settle in lowest energy state

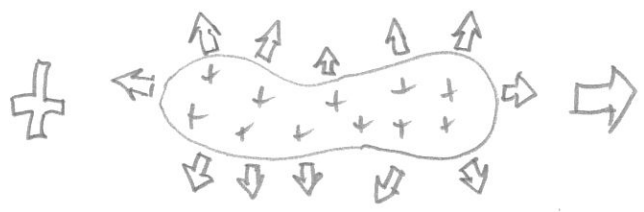


# Einstein's solution

Energy of moving body due to electric charge



Energy of its electromagnetic field



Energy associated with stresses induced by charges

$E_e =$

$$\frac{1}{8\pi} \int \frac{1}{\beta} \left[ x'^2 + \frac{1 + (v/c)^2}{1 - (v/c)^2} (y'^2 + z'^2) \right] d\xi d\eta d\zeta$$

Anisotropic

$\Delta E =$

$$-\frac{v^2}{c^2} \beta \frac{1}{4\pi} \int \xi x' \rho' d\xi d\eta d\zeta$$

$\frac{\partial x'}{\partial \xi} + \frac{\partial y'}{\partial \eta} + \frac{\partial z'}{\partial \zeta}$

Anisotropic

sum is isotropic!

$$E_e + \Delta E = \left[ \frac{1}{8\pi} \int (x'^2 + y'^2 + z'^2) d\xi d\eta d\zeta \right] \frac{1}{\sqrt{1 - v^2/c^2}}$$

Rest energy of electrostatic field  $E_s$

$$= \frac{\left( \frac{E_s}{c^2} \right) c^2}{\sqrt{1 - v^2/c^2}}$$

Electric charge add mass  $E_s/c^2$  that dilates with speed isotropically like ordinary mass

# Evaluating $E_e + \Delta E$

$$\Delta E = -\frac{v^2}{c^2} \beta \frac{1}{4\pi} \int_{\text{all space}} \left( \xi X' \left( \frac{\partial X'}{\partial \xi} + \frac{\partial Y'}{\partial \eta} + \frac{\partial Z'}{\partial \zeta} \right) \right) d\xi d\eta d\zeta$$

integrate by parts



$$\int_{\text{all space}} \left( \frac{\partial}{\partial \xi} \xi X'^2 + \frac{\partial}{\partial \eta} \xi X' Y' + \frac{\partial}{\partial \zeta} \xi X' Z' \right) - X'^2 - X' \frac{\partial X'}{\partial \xi} - Y' \frac{\partial X'}{\partial \eta} - Z' \frac{\partial X'}{\partial \zeta} d\xi d\eta d\zeta$$

By Gauss' theorem equal to an integral over an infinitely distant surface of vector

$$(\xi X'^2, \xi X' Y', \xi X' Z')$$

$\sim \frac{1}{\text{radius}^4}$  or  $\frac{1}{\text{radius}^3}$

$\therefore$  vanishes

$$-Y' \xi \frac{\partial^2 \phi}{\partial \xi \partial \eta}$$

$$= -Y' \xi \frac{\partial^2 \phi}{\partial \xi \partial \eta}$$

$$= Y' \xi \frac{\partial Y'}{\partial \xi}$$

$$Z' \xi \frac{\partial Z'}{\partial \xi}$$

$$(X', Y', Z') =$$

$$\left( -\frac{\partial \phi}{\partial \xi}, -\frac{\partial \phi}{\partial \eta}, -\frac{\partial \phi}{\partial \zeta} \right)$$

$$= \int_{\text{all space}} \left( -X'^2 - \xi X' \frac{\partial X'}{\partial \xi} - \xi Y' \frac{\partial Y'}{\partial \xi} - \xi Z' \frac{\partial Z'}{\partial \xi} \right) d\xi d\eta d\zeta$$

$$- \frac{1}{2} \xi \frac{\partial}{\partial \xi} (X'^2 + Y'^2 + Z'^2)$$

integrate by parts again

OVER

$$= \int_{\text{all space}} \left( -X'^2 - \frac{1}{2} \frac{\partial}{\partial \xi} \left( \xi (X'^2 + Y'^2 + Z'^2) + \frac{1}{2} (X'^2 + Y'^2 + Z'^2) \right) \frac{\partial \xi}{\partial \xi} \right) d\xi d\eta d\zeta$$

$$\int_{\text{all space}} \frac{\partial}{\partial \xi} \left( \xi (X'^2(\xi, \eta, \zeta) + \dots) \right) d\xi d\eta d\zeta$$

$$= \int_{\eta, \zeta} \left( \xi (X'^2(+\infty, \eta, \zeta) + \dots) - \xi (X'^2(-\infty, \eta, \zeta) + \dots) \right) d\eta d\zeta$$

$$= \int_{\eta, \zeta} 0 d\eta d\zeta = 0$$

$$= -\frac{1}{2} \int_{\text{all space}} (X'^2 - Y'^2 - Z'^2) d\xi d\eta d\zeta$$

Combining:

$$\Delta E = \frac{v^2}{c^2} \beta \frac{1}{8\pi} \int_{\text{all space}} (X'^2 - Y'^2 - Z'^2) d\xi d\eta d\zeta$$

$$E_e + \Delta E = \frac{1}{8\pi} \cdot \frac{1}{\beta} \left( x'^2 + \left[ \frac{1+(v/c)^2}{1-(v/c)^2} \right] (y'^2 + z'^2) \right) d\xi d\eta d\xi$$

$$+ \frac{v^2}{c^2} \frac{1}{8\pi} \beta \left( x'^2 - y'^2 - z'^2 \right) d\xi d\eta d\xi$$

$$\frac{1}{8\pi} \beta \int \frac{1}{\beta^2} \left( x'^2 + \left[ \frac{1+(v/c)^2}{1-(v/c)^2} \right] (y'^2 + z'^2) \right) d\xi d\eta d\xi$$

$$\uparrow$$

$$(1 - \frac{v^2}{c^2})$$

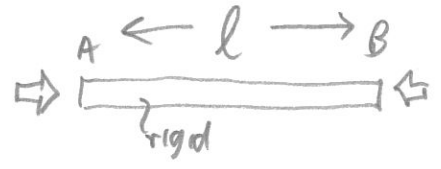
$$(1 - \frac{v^2}{c^2}) x'^2 + (1 + \frac{v^2}{c^2}) (y'^2 + z'^2) = (x'^2 + y'^2 + z'^2) - \frac{v^2}{c^2} (x'^2 - y'^2 - z'^2)$$

$$\therefore E_e + \Delta E = \beta \frac{1}{8\pi} \int (x'^2 + y'^2 + z'^2) d\xi d\eta d\xi$$

$$E_s$$

Impossibility of rigid rods/signals at  $v > c$

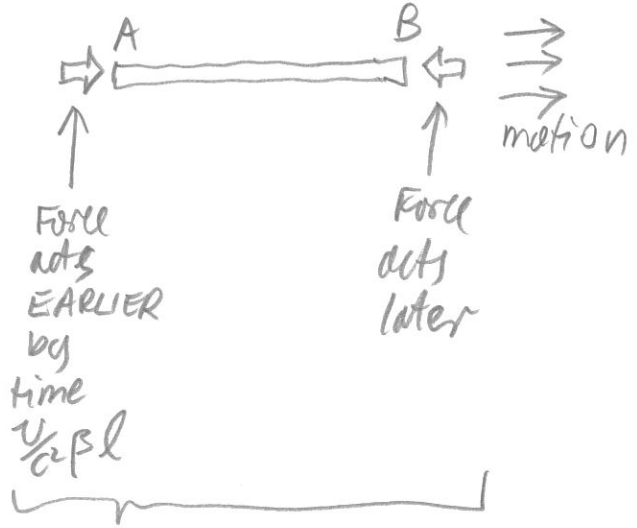
In rest frame of rod



Forces act for very short time at ends of rod

↓  
No change in rest state rod

In frame in which rod moves at  $v$



Rod's motion unchanged

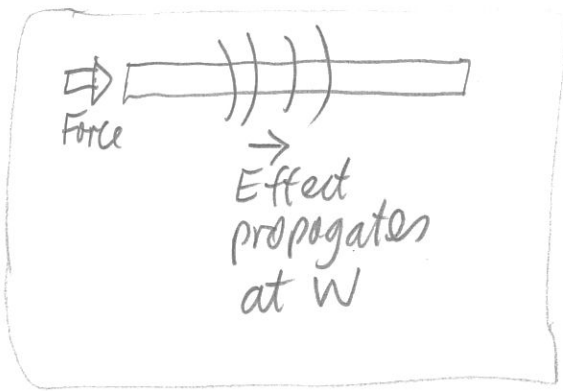
Force at A does work → supplies energy to rod BUT Rod's state does not change in any way.

∴ Energy of rod ≠ function (state of rod)

Hence rigidity untenable

(novel?) Dynamical argument

# Kinematic causal argument



view in frame that moves at +v

A stick figure representing an observer is shown moving to the right with velocity v. Below the figure is an arrow labeled v.

$$W' = \frac{W - v}{1 - \frac{Wv}{c^2}}$$

signal moves at W' covers distance l in time T

$$W' = \frac{l}{T} = \frac{W - v}{1 - \frac{Wv}{c^2}}$$

$$\therefore T = \frac{l \left(1 - \frac{Wv}{c^2}\right)}{W - v}$$

If  $W > c$

"effect... precedes cause (accompanied by act of will, for example"

"... does not contain a contradiction from a purely logical point of view ..."

conflicts so absolutely with the character of all experience, that the impossibility of the assumption ... proved ..."

$$= \frac{l}{W - v} \cdot \left(1 - \frac{W}{c} \cdot \frac{v}{c}\right)$$

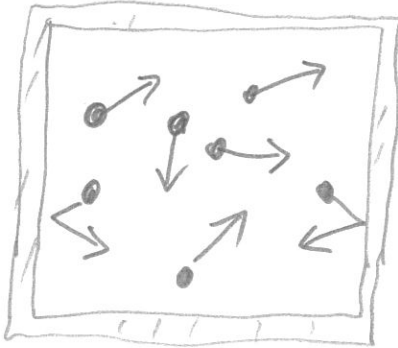
↑  
greater than 1

↑  
v < c can be brought close enough to c so that  $\left(1 - \frac{W}{c} \cdot \frac{v}{c}\right) < 0$

$$T < 0$$

Energy of a system consisting of mass points moving force-free

The obvious question.



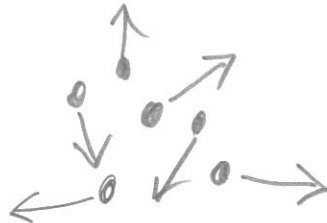
Each molecule of an ideal gas with rest mass  $\mu$  has total energy  $\mu c^2 \sqrt{1 - v^2/c^2}$

Does this <sup>total</sup> energy contribute to the rest mass of the gas system?

... Not quite the question Einstein answers.

Gas pressure  $\Rightarrow$  stress in chamber wall  $\Rightarrow$  Energy term  $\Delta E$  if gas + chamber moves as a whole.

Einstein just considers system of moving masses WITHOUT containment



$K(\xi, \eta, \zeta)$  is the rest frame in the sense that total momentum = 0



$$\sum \frac{m w_s}{\sqrt{1 - (w/c)^2}} = 0$$

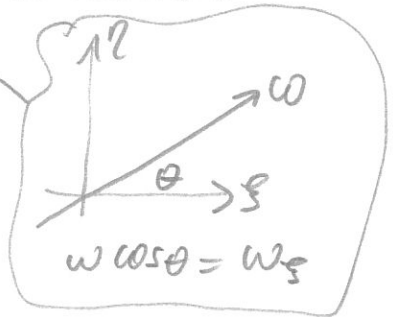
$$\sum \frac{m' w'_s}{\sqrt{1 - (w'/c)^2}} = 0$$

$$\sum \frac{m'' w''_s}{\sqrt{1 - (w''/c)^2}} = 0$$

In  $K(x, y, z)$  ...  $K$  moves at  $v$  in  $+x$  direction:

Energy  $E$  of each mass ("easily determined")

$$E = m c^2 \frac{1 + \frac{v w \cos \theta}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}} \sqrt{1 - \frac{w^2}{c^2}}}$$



$\therefore$  Total energy  $E = \sum_{\text{all masses}} E = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \sum m c^2 \cdot \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}} \right]$

(total energy/ $c^2$ ) in (momentum=0) frame = (rest mass of total system)

$$+ \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \sum \frac{m w \cos \theta}{\sqrt{1 - \frac{w^2}{c^2}}} \right]$$

0 by momentum condition

$$E = \frac{\left[ \sum \frac{m c^2}{\sqrt{1 - \frac{w^2}{c^2}}} \right] c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ***$$