The "Entwurf" Paper

§1 Equations of motion of the material point in the static gravitational field

Free fall in a static field

$$0 = \delta \int ds = \delta \int \sqrt{-dx^2 - dy^2 - dz^2 + c^2(x,y,z)dt^2}$$

$$= \delta \int \underbrace{\sqrt{c^2 - \dot{x}_i \dot{x}_i}}_{\text{"H"}}\, dt \qquad x_i = x_1, x_2, x_3 = x, y, z$$

summation of repeated i

where 1912 theory concluded

Euler-Lagrange equation

$$\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial H}{\partial \dot{x}_i} = 0 \quad \text{with:}$$

$$\frac{\partial H}{\partial x_i} = \frac{\partial}{\partial x_i}(c^2 - \dot{x}_i\dot{x}_i)^{\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{c^2 - \dot{x}_i\dot{x}_i}} \cdot \frac{\partial c^2}{\partial x_i} = \frac{c\,\partial c/\partial x_i}{\sqrt{c^2 - \dot{x}_i\dot{x}_i}}$$

$$\frac{\partial H}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}(c^2 - \dot{x}_k\dot{x}_k)^{\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{}}\frac{\partial \dot{x}_k\dot{x}_k}{\partial \dot{x}_i} = \frac{\dot{x}_i}{\sqrt{c^2 - \dot{x}_k\dot{x}_k}}$$

add m

$$\frac{d}{dt}\left\{\frac{m\dot{x}_i}{\sqrt{c^2 - \dot{x}_k\dot{x}_k}}\right\} = -\underbrace{\frac{mc\,\partial c/\partial x_i}{\sqrt{c^2 - \dot{x}_k\dot{x}_k}}}_{\text{Gravitational force}}$$

canonical momentum $J_i = m\frac{\partial H}{\partial \dot{x}_i} = \frac{m\dot{x}_i}{\sqrt{c^2 - \dot{x}_k\dot{x}_k}} \approx \frac{m\dot{x}_i}{c}$

small velocities

conserved Hamiltonian

$$E = J_i \dot{x}_i - mH$$

$$\boxed{E = \frac{m\dot{x}_i\dot{x}_i}{\sqrt{c^2 - \dot{x}_k\dot{x}_k}} - \underbrace{\frac{m(c^2 - \dot{x}_i\dot{x}_i)}{\sqrt{c^2 - \dot{x}_k\dot{x}_k}}}_{\text{same as } m\sqrt{c^2 - \dot{x}_k\dot{x}_k}} = \frac{mc^2}{\sqrt{c^2 - \dot{x}_k\dot{x}_k}} \approx mc + \frac{1}{2}m\dot{x}_k\dot{x}_k}$$

small velocities

§2 Equations of motion ... arbitrary gravitational field

Particle in free fall follows

$$\delta\int ds = 0 \quad \text{where}$$

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx_\mu dx_\nu$$

$g_{\mu\nu}$: "fundamental tensor"

standard analysis in Lagrangian mechanics (as before)

canonical momentum

$$J_x = -\sum_\nu m\, g_{1\nu} \frac{dx_\nu}{ds}$$

x_1, x_2, x_3, x_4
$\nu = 1,2,3,4$

Force

$$K_x = -\frac{1}{2} m \sum_{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_1} \frac{dx_\mu}{ds} \cdot \frac{dx_\nu}{dt}$$

dt: x_4

Special case of static field

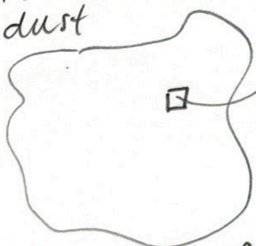
$$g_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & c^2(x,y,z) \end{bmatrix}$$

$$J_x = -\sum_\nu m g_{1\nu} \frac{dx_\nu}{ds} = -\sum_\nu m g_{1\nu} \underbrace{\frac{dx_\nu}{dt}}_{-\frac{dx_1}{dt}} \cdot \underbrace{\frac{dt}{ds}}_{\frac{1}{\sqrt{c^2-q^2}}} = \frac{m \dot{x}_1}{\sqrt{c^2-q^2}}$$

As before

$$K_x = -\frac{1}{2} m \underbrace{\frac{\partial g_{44}}{\partial x_1}}_{\text{all others constant}} \underbrace{\frac{dx_4}{ds}}_{\frac{1}{\frac{ds}{dt}}} \cdot \underbrace{\frac{dx_4}{dt}}_{\frac{dt}{dt}=1} = -\frac{1}{2} m \frac{\partial c^2/\partial x_1}{\sqrt{c^2-q^2}} = -mc \frac{\partial c/\partial x}{\sqrt{c^2-q^2}}$$

§4 Motion of Continuously Distributed Incoherent Masses



From section §2, we have expressions for momentum J
Force K
Energy E
& Equations of motion

Re-express these equations in terms of densities of momentum, energy, force

$$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\, g_{\sigma\mu}\, \Theta_{\mu\nu}\right) - \frac{1}{2}\sum_{\mu\nu} \sqrt{-g}\, \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\, \Theta_{\mu\nu} = 0$$

accumulation energy-momentum

gravitational force density

where $\Theta_{\mu\nu} = \rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds}$

contravariant stress-energy tensor
covariant form is $T_{\mu\nu}$
i.e. Einstein "raises" indices
$T \longrightarrow \Theta$

x-momentum density is
$-\sqrt{-g}\, g_{1\nu}\, \Theta_{\nu 4}$
Energy density is
$-\sqrt{-g}\, g_{4\nu}\, \Theta_{\nu 4}$

Law of Energy-momentum conservation

- Grossmann. Math Section. §4: Above law corresponds to covariant divergence $\Theta_{\mu\nu} = 0$ ... special form possible since $\Theta_{\mu\nu} = \Theta_{\nu\mu}$ (symmetric) — Natural generalization of result in special relativity
- Generalize law:
  For all systems, this is the law of energy-momentum conservation.

Reminder: Stress-Energy Tensor in Special Relativity

$\boxed{c=1}$

In any medium

Energy conservation

$$\underbrace{\frac{\partial W}{\partial t}}_{\text{energy density}} + \nabla \cdot \underbrace{\underline{S}}_{\text{energy flux}} = 0$$

momentum conservation

$$\underbrace{\frac{\partial \underline{g}}{\partial t}}_{\text{momentum density}} + \nabla \cdot \underbrace{\underline{\underline{t}}}_{\text{stress tensor of medium}} = 0$$

$$\frac{\partial W}{\partial t} + \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z} = 0$$

$$\frac{\partial g_x}{\partial t} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} = 0$$

$$\frac{\partial g_y}{\partial t} + \frac{\partial t_{yx}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z} = 0$$

$$\frac{\partial g_z}{\partial t} + \frac{\partial t_{zx}}{\partial x} + \frac{\partial t_{zy}}{\partial y} + \frac{\partial t_{zz}}{\partial z} = 0$$

$$\Longrightarrow \quad \frac{\partial T_{ik}}{\partial x_k} = 0$$

$x_k = t, x, y, z$

$$T_{ik} = \left[\begin{array}{c|ccc} W & \multicolumn{3}{c}{\overbrace{S_x \quad S_y \quad S_z}^{\text{energy flux}}} \\ \hline g_x & t_{xx} & t_{xy} & t_{xz} \\ g_y & t_{yx} & t_{yy} & t_{yz} \\ g_z & t_{zx} & t_{zy} & t_{zz} \end{array}\right]$$

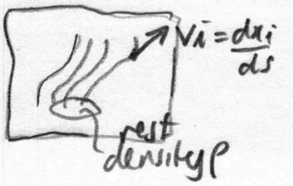
momentum density: g_x, g_y, g_z

Symmetry of T_{ik} expresses version of $E = mc^2$:

momentum density $\underline{g}$ = energy flux $\underline{S}$

Simplest stress energy tensor for dust

dust trajectories



$$T_{ik} = \rho V_i V_k$$

In momentary rest frame of dust, velocity $= 0 \Rightarrow V_i = (1,0,0,0)$

$$T_{ik} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

§5 The Differential Equations of the Gravitational Field

The Problem posed:

Poisson's equation in Newtonian theory
$\Delta\varphi = 4\pi k\rho$

$\xrightarrow{\text{generalize}}$

Equations that determine how gravitational field $g_{\mu\nu}$ generated by sources & spread through space

ten components in $g_{\mu\nu}$ ∴ seek 10 equations i.e. second rank tensor equations

$$\kappa\,\Theta_{\mu\nu} = \Gamma_{\mu\nu}$$

stress-energy tensor natural mass source term

"Gravitation tensor" constructed from $g_{\mu\nu}$, $\frac{\partial g_{\mu\nu}}{\partial x_\nu}$, $\underbrace{\frac{\partial^2 g_{\mu\nu}}{\partial x_1 \partial x_2}}_{\text{2nd derivatives}}$

Analogous to $\Delta\varphi$

Form expected for $\Gamma_{\mu\nu}$:

New "$g^{\mu\nu}$"

$$\boxed{\Gamma_{\mu\nu} = \underbrace{\sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right)} + \underbrace{\text{terms in } \frac{\partial \gamma_{\mu\nu}}{\partial x_\alpha} \text{ that vanish in weak field}}}$$

"Core form"

$g_{\mu\nu} \approx \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & c^2 \end{bmatrix}$ weak field

In weak field, $g_{\mu\nu}$ differs from in terms that are ε small

so terms like

$$\sum_{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial x_\mu}\cdot\frac{\partial \gamma_{\alpha\beta}}{\partial x_\nu} \sim \varepsilon^2$$

are expected

$$\underbrace{-\frac{\partial^2\gamma_{\mu\nu}}{\partial x_1^2} - \frac{\partial^2\gamma_{\mu\nu}}{\partial x_2^2} - \frac{\partial^2\gamma_{\mu\nu}}{\partial x_3^2} + \frac{1}{c^2}\frac{\partial^2\gamma_{\mu\nu}}{\partial x_4^2}}$$

In static fields only non-zero term is $\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\gamma_{44}$

$$= \Delta(c^2) \approx 2c\,\Delta c$$

Einstein cannot find a generally tensor for $\Gamma_{\mu\nu}$!

Attempts:

(1) Form derivatives of $g_{\mu\nu}$ using covariant derivative $\Longrightarrow$ $\underbrace{\text{covariant derivative } g_{\mu\nu} \equiv 0}$

condition using it will place no constraint on $g_{\mu\nu}$

(2) Grossmann (mathematical Part. §4.2)

Build $\Gamma_{\mu\nu}$ from "Riemann differential tensor"

$$R_{iklm} = (ik,lm) = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x_k \partial x_l} + \frac{\partial^2 g_{kl}}{\partial x_i \partial x_m} - \frac{\partial^2 g_{il}}{\partial x_k \partial x_m} - \frac{\partial^2 g_{mk}}{\partial x_l \partial x_i}\right) + \sum_{\rho\sigma} \gamma_{\rho\sigma}\left(\begin{bmatrix} im \\ \rho \end{bmatrix}\begin{bmatrix} kl \\ \sigma \end{bmatrix} - \begin{bmatrix} il \\ \rho \end{bmatrix}\begin{bmatrix} km \\ \sigma \end{bmatrix}\right)$$

$\Big\Downarrow$ contract over 2 indices

$$G_{im} = \sum_k \{ik, km\}$$

Now called the "Ricci tensor"
$G_{im} = 0$ are the source free equations of the final theory

"It turns out, however, that in the special case of infinitely weak, static gravitational fields this tensor does not reduce to the expression $\Delta\varphi$."

?

What went wrong?
Standard view circa 1980 (e.g. Pais biography)

$$G_{im} = \frac{1}{2}\gamma^{kl}\left[\frac{\partial^2 g_{im}}{\partial x_k \partial x_l} + \frac{\partial^2 g_{kl}}{\partial x_i \partial x_m} - \frac{\partial^2 g_{il}}{\partial x_k \partial x_m} - \frac{\partial^2 g_{mk}}{\partial x_l \partial x_i}\right] + \text{Terms quadratic in } \frac{\partial g_{\mu\nu}}{\partial x_\alpha}$$

Vanish as ε^2 small in weak field

weak field

$$\Sigma\left(\frac{\partial^2}{\partial x_i^2} + \cdots\right) g_{im}$$

$$\downarrow$$

$$\Delta\varphi$$

How can we eliminate these terms so G_{im} reduces to $\Delta\varphi$?

Expression valid for ALL coordinate systems

Expression valid only in linearly related coordinate systems

so reduction can ONLY succeed if we restrict our coordinate systems.

(New) standard device = specify four conditions to fix coordinate system

plausibility. coordinate system = 4 arbitrary scalar fields.

4 conditions use up freedom in this arbitrariness

e.g. "Harmonic condition" $\Longrightarrow$ choice of harmonic coordinates

$$\gamma_{kl}\begin{bmatrix} kl \\ i \end{bmatrix} = 0$$

3 second derivative terms vanish

Supposition: Einstein did not recognize this freedom!

$\boxed{\text{Stachel: Perhaps not so simple! 1979}}$

Grossman reports failure of G_{im} for weak STATIC fields:

$$\boxed{\begin{matrix} G_{im} \\ =0 \end{matrix}} \xrightarrow{\text{solve for weak static field of Einstein's 1913 theory}} \boxed{\begin{aligned} G_{44} &= \tfrac{1}{2}\Delta\varphi = 0 \\ G_{ij} &= -\frac{1}{2}\frac{\partial^2\varphi}{\partial x_i \partial x_j} = 0 \quad i,j = 1,2,3 \end{aligned}}$$

$$\underbrace{g_{\mu\nu} \doteq \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1+\varphi \end{bmatrix}}_{\text{Natural assumption}}$$

Disaster!
φ varies as linear function of coordinates only.

- Principle of equivalence
- Solution to $\Box g_{\mu\nu} = kT_{\mu\nu}$ for static dust cloud
- g_{44} only enters in equation of motion of a mass point in weak field.

BUT... in final theory, weak fields are

$$g_{\mu\nu} \approx \begin{bmatrix} -1-\varphi & & & 0 \\ & -1-\varphi & & \\ & & -1-\varphi & \\ 0 & & & 1+\varphi \end{bmatrix}$$

i.e. weak static fields are <u>not</u> spatially flat.

Einstein's solution : Give up. Constraint of energy-momentum conservation already all but fixes field equations. Leave covariance open!

Lesson of 1912

Einstein illustrates his method : Trivial case of electrostatics

Field equation $\partial_i \partial_i \varphi = \rho$, $\left(\partial_i = \frac{\partial}{\partial x_i}\right)$

substitute

Force density on charges $= (\partial_i \varphi)\rho = \partial_i \varphi (\partial_k \partial_k \varphi) = \underbrace{\partial_k\left(\partial_i \varphi \partial_k \varphi - \frac{1}{2}\delta_{ik}(\partial_m \varphi \partial_m \varphi)\right)}$

Divergence of stress tensor. Must get this else action/reaction violated --- Lesson of 1912

This much is just an identity. Existence of stress tensor complace no further constraint on φ after field equation fixed!

DISCOVER field equation by reversing calculation.

(1) Find identity in φ of form.

$$\partial_i \varphi \left(\begin{matrix}\text{something}\\ \text{in } \partial_k \partial_k \varphi\end{matrix}\right) = \partial_k \left(\begin{matrix}\text{something in } \partial_k \varphi \\ \text{that looks like stress tensor}\end{matrix}\right) \text{ e.g.}$$

(2) Read off Field term $= \rho$ in field equation

$\partial_k \partial_k \varphi$ is we use the above identity!

Einstein generates the Entwurf Gravitational Field Equations

seek identity of the form :

$$\frac{\partial}{\partial x_\alpha}\left(\begin{array}{l}\text{something that looks}\\ \text{like a grav. field}\\ \text{stress tensor}\end{array}\right) = \overbrace{\frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\,\frac{\partial g_{\mu\nu}}{\partial x_\sigma}}^{\text{From force density term in energy-momentum conservation law}}\left(\begin{array}{l}\text{something in 1st, 2nd}\\ \text{derivatives of } g_{\mu\nu}\\ \text{of core form}\end{array}\right)$$

Grossmann (mathematical Part §4) supplies identity:

$$\left.\sum_{\alpha\beta\tau\rho}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\,\frac{\partial \gamma_{\tau\rho}}{\partial x_\beta}\cdot\frac{\partial g_{\tau\rho}}{\partial x_\sigma}\right) - \frac{1}{2}\sum_{\alpha\beta\tau\rho}\frac{\partial}{\partial x_\sigma}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\,\frac{\partial \gamma_{\tau\rho}}{\partial x_\alpha}\cdot\frac{\partial g_{\tau\rho}}{\partial x_\beta}\right)\right\}\text{Read off stress-ene}$$

$$= \sum_{\mu\nu}\sqrt{-g}\,\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\left\{\sum_{\alpha\beta}\frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{-g}\,\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right) - \sum_{\alpha\beta\tau\rho}\gamma_{\alpha\beta}\,g_{\tau\rho}\,\frac{\partial \gamma_{\mu\tau}}{\partial x_\alpha}\,\frac{\partial \gamma_{\nu\rho}}{\partial x_\beta}\right.$$

$$\left.\underbrace{+\frac{1}{2}\sum_{\alpha\beta\tau\rho}\gamma_{\alpha\mu}\gamma_{\beta\nu}\,\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\,\frac{\partial \gamma_{\tau\rho}}{\partial x_\beta} - \frac{1}{4}\sum_{\alpha\beta\tau\rho}\gamma_{\mu\nu}\gamma_{\alpha\beta}\,\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\,\frac{\partial \gamma_{\tau\rho}}{\partial x_\beta}}_{\text{This is } 2\Gamma_{\mu\nu}}\right\}$$