Heisenberg's "Umdeutung..."

What I expected.

Old quantum theory [differential → difference operators; 1 index quantities → two index quantities; DONE ONCE ⟹ Protoversion of new matrix mechanics

What I found

Old quantum theory analysis of general & particular systems [⋮ as above ⋮ DONE REPEATEDLY ⟹ Proto-matrix mechanical analysis of each system

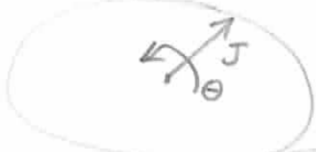
Hence old quantum principles like the correspondence principle are still invoked.

Heisenberg's "Umdeutung" paper	The overall project.

Old Quantum theory		Heisenberg
System described by classical equations of motion $\ddot{x} + f(x) = 0$	KEEP $\longrightarrow$	$\ddot{x} + f(x) = 0$
Fourier expansion $x(t) = \sum A(n) e^{i\omega(n)t}$	REPLACE $\longrightarrow$	substitute into; set of two index quantities $\{A(n,\alpha) e^{i\omega(n,\alpha)t}\}$
(old) Quantum condition: Allowed orbits satisfy $nh = J = \oint p\,dq$	REPLACE by converting d/dn to difference $\longrightarrow$	New quantum condition on $A(n,\alpha), \omega(n,\alpha)$
Correspondence principle uses $A(n)$ to infer which quantum transitions are possible	$- - - - - - \rightarrow$	still tacitly employed?

Differential operator $\longrightarrow$ Difference operator

Classical orbits



Hamilton's equations

$\dot{q} = \frac{\partial H}{\partial p}$ $\dot{p} = -\frac{\partial H}{\partial q}$

select action, angle canonical variables

$q \leftrightarrow \theta$

$p \leftrightarrow J = \oint p\,dq$

$H \leftrightarrow W$

$\dot{J} = -\frac{\partial W}{\partial q} = 0$ since radial symmetry W

$J = \text{constant}$

$\dot{\theta} = \frac{\partial W}{\partial J} = \nu \leftarrow$ angular velocity.

(old)

Quantize | Allow only orbits with $J = nh$

$\nu = \frac{\partial W}{\partial J} = \frac{1}{h}\frac{\partial W}{\partial n}$

For large n, $\frac{\partial}{\partial n}$ makes sense

$\nu \Leftrightarrow$ lowest order frequency emitted waves

In general, have all harmonics

$\nu(n,\alpha) = \alpha\,\nu(n)$ $\alpha = 1, 2, 3, \ldots$

$$\boxed{\nu(n,\alpha) = \alpha\,\nu(n) = \frac{\alpha}{h}\frac{\partial W}{\partial n}}$$

Discretize

$W(n-\alpha) = W(n) - \alpha\frac{\partial W(n)}{\partial n} + \ldots$

Question:
How unique is the discretization?

$$\boxed{\nu(n, n-\alpha) = \frac{1}{h}\, W(n) - W(n-\alpha)}$$

Found empirically for relation emitted frequencies ν and energies of orbits W

Hence infer that we should discretize.

$\boxed{\text{Differential Operator} \longrightarrow \text{Difference operator} \quad \text{using new representation}}$

$$J = \oint p\,dq = \oint m\dot{x}\,dx = \oint m\dot{x}\underbrace{\frac{dx}{dt}}_{\dot{x}}dt = \oint m\dot{x}^2\,dt$$

$$\Downarrow$$

$$x = \sum_{-\infty}^{\infty}{}_{\alpha}\, a_\alpha(n)\, e^{i\alpha w_n t}$$

$$m\dot{x} = m\sum_{-\infty}^{\infty}{}_{\alpha}\, a_\alpha(n)\, i\alpha w_n\, e^{i\alpha w_n t}$$

$a_{-\alpha}(n) = \overline{a_\alpha(n)}$ since x is real

$$J = 2\pi m \sum_{-\infty}^{\infty}{}_{\alpha} |a_\alpha(n)|^2 \alpha^2 w_n = nh$$

$$\Downarrow d/dn$$

$$h = 2\pi m \sum_{-\infty}^{\infty}{}_{\alpha}\, \alpha \frac{d}{dn}\big(\underbrace{\alpha w_n}_{=\nu(n,\alpha)} |a_\alpha|^2\big)$$

same as $\underbrace{\nu(n,\alpha)}_{\alpha w_n} = \frac{\alpha}{h}\frac{\partial W}{\partial n}$

Introduce difference operators, two index quantities

$\alpha \frac{d f(n)}{dn} = f(n) - f(n-\alpha)$

$\nu(n+\alpha, n) = w(n+\alpha, n)$ $\quad a(n+\alpha, n)$

NB Could have use $\nu(n, n-\alpha)$ here. But we would get a different formula that is equivalent under relabelling

$$2\pi m \sum_{-\infty}^{\infty} w(n+\alpha,n)|a(n+\alpha,n)|^2 - \underbrace{w(n-\alpha+\alpha, n-\alpha)\,|a(n-\alpha+\alpha, n-\alpha)|^2}_{w(n,n-\alpha)\,|a(n,n-\alpha)|^2}$$

$$\Downarrow a_{-\alpha}(n) = \bar{a}_\alpha(n)$$

$$\boxed{4\pi m \sum_{\alpha=0}^{\infty} w(n+\alpha,n)\,|a(n+\alpha,n)|^2 - w(n,n-\alpha)\,|a(n,n-\alpha)|^2}$$

$\boxed{\text{H's } 16}$

NB: Heisenbergh has $w(n,n+\alpha)$, $a(n,n+\alpha)$] other cases DO NOT have this typo/inversion

Plausibly $w(n,n+\alpha) = w(n+\alpha,n)$.

But can we have $a(n,n+\alpha) = a(n+\alpha,n)$??

Process | Heisenberg does not apply this $\binom{H}{16}$ in each application. INSTEAD...

He ① Writes down the old quantum differential expression (in case of rotor, after applying Bohr's correspondence principle)

② Redoes the discretization.

Hence Heisenberg's process is more dependent on the older theory than we'd expect

Example: Anharmonic oscillator.

p. 271 Heisenberg writes the old quantum

$$1 = 2\pi m \frac{d}{dJ} \sum_{-\infty}^{\infty} \frac{1}{4} \tau^2 |a_\tau| \omega$$

This factor not present earlier. Classical source ??

$\Downarrow$ Discretize

$$h = \pi m \sum_{0}^{\infty} \left[|a(n+\tau,n)|^2 \omega(n+\tau,n) - |a(n,n-\tau)|^2 \omega(n,n-\tau) \right)$$

Differs from $\binom{H}{16}$. No factor of 4

Now in the correct order.

Example: Heisenberg's Rotor

"electron describes a plane, uniform rotation at a distance a with angular velocity ω"

Action

$J = \oint p\, dq = 2\pi m a^2 \omega$

angular momentum $ma^2\omega$

θ

Old quantum condition in terms of differential operators is:

$$nh = J = 2\pi m a^2 \omega$$

$$h = \frac{dJ}{dn} = \frac{d}{dn}(2\pi m a^2 \omega)$$

Discretize

$$\omega(n) \longrightarrow \omega(n+1, n)$$

$$\frac{d}{dn} f(n) = f(n) - f(n-1)$$

$$h = 2\pi m \left\{ a^2\, \omega(n+1, n) - a^2\, \omega(n-1+1, n-1) \right\}$$

$$= 2\pi m \left\{ a^2 \omega(n+1, n) - a^2 \omega(n, n-1) \right\}$$

Note: ① Heisenberg has chosen as part of this discretization <u>not</u> to replace a by a two-index representative.
WHY?
... working backwords from Zeeman Effect ???

② The old quantum condition already assumes that one step transitions $n \to n-1$, etc. are the <u>only</u> possible ones.
Hence <u>NO</u> α's or τ's here
Bohr's correspondence principle used to arrive at this?

Apply Bohr correspondence principle to the rotor

$\underline{r} = (x, y)$

$|\underline{r}| = R$

$x(t) = R\cos\omega t$
$y(t) = R\sin\omega t$

$$x(t) = R\cos\omega t + 0\cos 2\omega t + \cdots$$
$$y(t) = \underbrace{R\sin\omega t}_{\text{one step transitions possible}} + \underbrace{0\sin 2\omega t + \cdots}_{\text{Two step and higher are excluded.}}$$

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Solve $h = 2\pi m a^2 \{ \omega(n+1,n) - \omega(n,n-1) \}$

For ground state $n_0 = 0$, $\omega(n_0, n_0 - 1) = 0$

Solving Rotor

$\therefore \quad h = 2\pi m a^2 \omega(1,0) \Rightarrow \omega(1,0) = \frac{h}{2\pi m a^2}$

$\therefore \quad h = 2\pi m a^2 \{ \omega(2,1) - \omega(1,0) \} \Rightarrow \omega(2,1) = \frac{2h}{2\pi m a^2}$

$\therefore \quad h = 2\pi m a^2 \{ \omega(3,2) - \omega(2,1) \} \Rightarrow \omega(3,2) = \frac{3h}{2\pi m a^2}$

$\vdots$

$$\boxed{\omega(n,n-1) = \frac{nh}{2\pi m a^2}}$$

Heisenberg says he used his (7),(8), but I don't see how.

classical

$$\boxed{W = \frac{1}{2} m v^2}$$

HOW ??

$v^2 = a^2 \omega^2$??
use representative $\omega(n+1,n)^2$ or $\omega(n,n-1)^2$??
which ?... Take average ??

$$W = \frac{m}{2} a^2 \left\{ \frac{\omega^2(n,n-1) + \omega^2(n+1,n)}{2} \right\} = \frac{ma^2}{4} \left[\frac{n^2 h^2}{4\pi^2 m^2 a^4} + \frac{(n+1)^2 h^2}{4\pi^2 m^2 a^4} \right]$$

$$= \frac{h^2}{16\pi^2 m a^2} \underbrace{\left[n^2 + (n+1)^2 \right]}_{2n^2 + 2n + 1} = \frac{h^2}{8\pi^2 m a^2} \left[n^2 + n + \frac{1}{2} \right]$$

$$W(n)-W(n-1) = \frac{h^2}{8\pi^2 ma^2}\left[n^2+n+\cancel{\tfrac{1}{2}} -(n-1)^2-(n-1)-\cancel{\tfrac{1}{2}}\right]$$

$$= \frac{h^2}{8\pi^2 ma^2}\left[n^2-n^2+2n-1+1\right] = \frac{h^2 n}{4\pi^2 ma^2}$$

$$= \frac{h}{2\pi}\cdot\underbrace{\frac{hn}{2\pi ma^2}}_{\omega(n,n-1)}$$

$$\therefore \boxed{\omega(n,n-1) = \frac{2\pi}{h}\left[W(n)-W(n-1)\right]}$$