Optical mechanical Analogies

Wave propagation in media of non-constant refractive index n

General form of scalar wave

vector position

"eikonal" or "optical path length"

$$\phi = \exp\left[A(\underline{x}) + ik_0 \left(L(x) - ct \right) \right]$$

L encodes effect of variations in n.

Take special case of n = constant

$L(\underline{r}) = nz$ for plane waves propagating in z-direction

$$\phi = \exp\left(ik_0(nz - ct)\right)$$

Solve for wave speed: Points of constant phase satisfy $nz - ct = \text{const}$

$\therefore\ z = \frac{c}{n} t + \text{const}$

wave speed = c/n

Goldstein, classical mechanics pp. 488-9

After some vector analysis, Assume n varies slowly on spatial scale set by wavelength

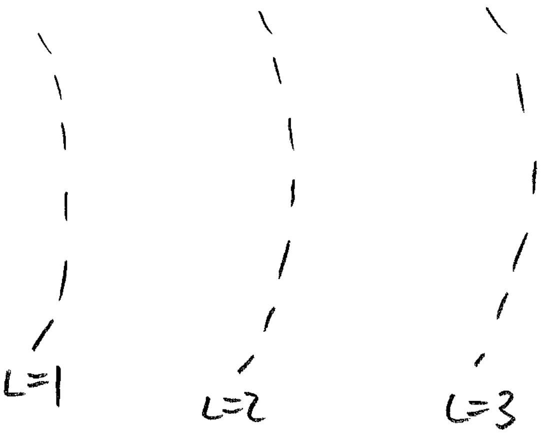
Eikonal Equation $\quad (\nabla L)^2 = n^2$

special case
$L(\underline{r}) = nz$
$|\nabla L| = n$
$|\nabla L|^2 = n^2$

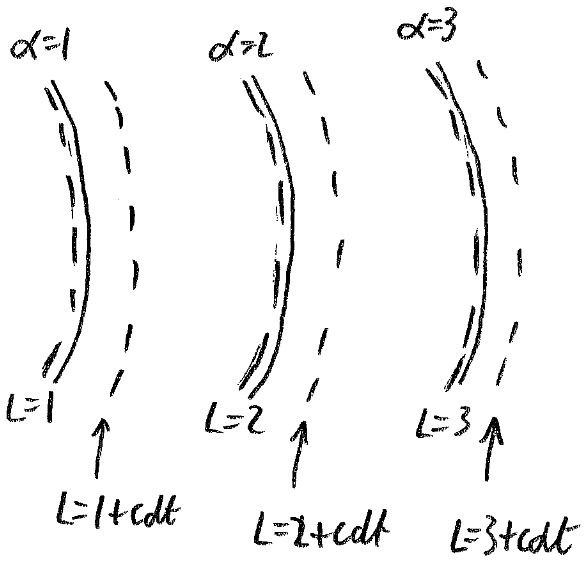
Goldstein, Classical mechanics

Visualizing wave propagation due to term $\exp ik_0(\underbrace{L(x)-ct}_{\text{phase } \alpha(t)})$

$L(\underset{\sim}{r})$ is independent of time and maps out a background field over which wave propagates

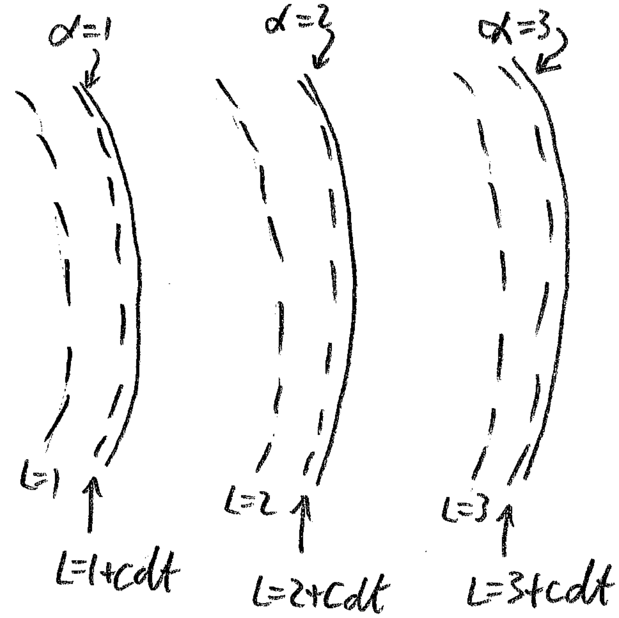


At, $t=0$, level surfaces $\alpha(t), L(t)$ co-incide



$\Rightarrow$

At $t=0+dt$, level surfaces of α have moved!

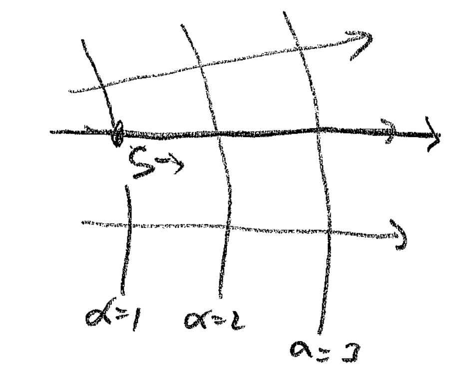


$$\alpha(0)=L(\underline{r})-c.0=L(r)$$

$$\alpha(dt)=L(\underline{r})-c.dt$$

Quick & Dirty Version of the Eikonal Equation

$\alpha(t)$ phase propogates



choose path orthogonal to surfaces of constant α

S is the path length

Pick some definite value of $\alpha = \bar{\alpha}$. We track the speed of this value in the propogation.

It will be at $S_{\bar{\alpha}}(t)$ on the path

That is, $\bar{\alpha} = L(S_{\bar{\alpha}}(t)) - ct) = \text{constant}$

fixed value
∴ no t dependence.

$$\therefore\ 0 = \frac{d}{dt}\left[L(S_{\bar{\alpha}}(t)) - ct\right] = \underbrace{\frac{dL}{dS}}_{|\nabla L|} \cdot \underbrace{\frac{dS_{\bar{\alpha}}(t)}{dt}}_{\text{speed } u \text{ of wave}} - c$$

$$\therefore\ \boxed{u = \frac{c}{|\nabla L|}}$$

[Define n by $u = c/n$

$$\Rightarrow \boxed{|\nabla L| = n \quad \text{or} \quad |\nabla L|^2 = n^2}$$

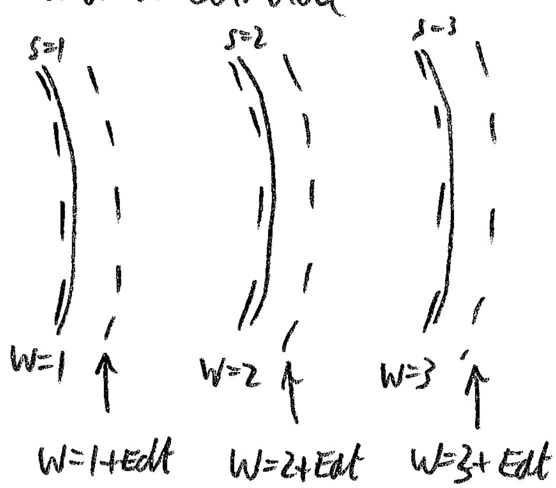
For time independent Hamiltonian

Time development of $S(t)$ is analogous to time development of wave phase $\alpha(t)$.

we have

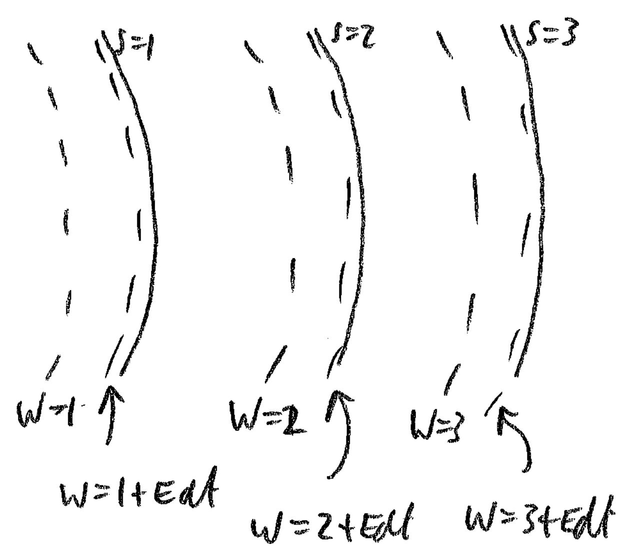
$$\underset{\boxed{S=\int L\,dt \text{ + constant}}}{S}(q_i, \alpha_i, t) = \underbrace{W}_{\substack{\text{maps out} \\ \text{time independent} \\ \text{field over which} \\ S \text{ propagates}}}(q_i, \underset{\substack{\text{constants} \\ \text{of motion}}}{\alpha_i}) - Et$$

At $t=0$, level surfaces S and W coincide



$$S(0) = W + E.0 = W$$

A $t=0+dt$, level surfaces of S have moved



$$S(dt) = W + E.dt$$

copy earlier analysis of Eikonal equation

speed of wave in configuration space

$$u = \frac{ds}{dt} = \frac{E}{|\nabla W|}$$

Special case of Lagrangian

$$L = T - V$$

T: Quadratic in $\dot{q}_i$

$V = V(q_i)$

Induces new "metric" in which all distances are measured

IMPORTANT

$$\sum_{ik} A_{ik}\dot{q}_i\dot{q}_k$$

special case: $T = \frac{1}{2}mv^2$

Hence $p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\sum A_{mn}\dot{q}_m\dot{q}_n = \sum_k A_{ik}\dot{q}_k + \sum_k A_{ki}\dot{q}_k$

$$= \sum_k (A_{ik}+A_{ki})\dot{q}_k$$

$$H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \underbrace{\sum_{ik}(A_{ik}+A_{ki})\dot{q}_k\dot{q}_i}_{2T} - (T-V) = T+V$$

Common case: $A_{ik} = \frac{m}{2}\delta_{ik}$ $\quad p_i = \sum_k \underbrace{(A_{ik}+A_{ki})}_{m\delta_{ik}}\dot{q}_k = m\dot{q}_i$

More generally, use $(A_{ik}+A_{ki})$ as a "metric"

Define $|p_i|^2 = \sum_{ik}(A_{ik}+A_{ki})\dot{q}_k\dot{q}_i = \sum_i p_i\dot{q}_i$

so that $T = \frac{1}{2}|p_i|^2$

Relation to Action Principles

$$H = T + V \leftarrow V = V(q_i)$$

H: constant E

T: Quadratic in $\dot{q}_i$

$\therefore$ From earlier

$$T = \frac{1}{2}|p_i|^2 = \frac{1}{2}\underbrace{\left|\frac{\partial W}{\partial \dot{q}_i}\right|^2}_{\text{“}|\nabla W|\text{”}}$$

From Hamilton-Jacobi theory $p_i = \dfrac{\partial W}{\partial \dot{q}_i}$

$|\;|$ uses generalized metric

s = distance in configuration space using new metric

Time to pass from point P_1 to point P_2 in config. space

$$= \int_{P_1}^{P_2} dt = \int_{P_1}^{P_2} \frac{ds}{u} = \int_{P_1}^{P_2} \frac{|\nabla W|}{E} ds = \int_{P_1}^{P_2} \frac{\sqrt{2T}}{E} ds = \int_{t_1}^{t_2} \frac{2T}{E} dt$$

$u = ds/dt$ $\qquad$ $\frac{1}{u} = \frac{|\nabla W|}{E}$

$$= \frac{1}{E}\int_{t_1}^{t_2} 2T\, dt$$

constant of motion

in the config. space metric:

$$2T = \sum_{ik} (A_{ik} + A_{ki})\dot{q}_i \dot{q}_k = \left|\frac{ds}{dt}\right|^2$$

since $ds^2 = \sum_{ik} (A_{ik} + A_{ki}) dq_i\, dq_k$

$$\therefore \sqrt{2T} = \left|\frac{ds}{dt}\right|$$

$$\therefore \sqrt{2T}\, ds = \left|\frac{ds}{dt}\right| \cdot ds = \left|\frac{ds}{dt}\right| \cdot \left|\frac{ds}{dt}\right| \cdot dt = 2T\, dt$$

That is:

$$\underbrace{\begin{matrix}\text{Time to}\\ \text{pass from}\\ \text{point 1}\\ \text{to Point 2}\end{matrix}}_{\text{seek motion that MINIMIZES}} = \underbrace{\int_{P_1}^{P_2} \frac{ds}{u}}_{} = \underbrace{\frac{1}{E}\int_{t_1}^{t_2} 2T\, dt}_{}$$

minimize time of motion is Fermat's principle in optics

$$\frac{1}{u} = \frac{n}{c_{\leftarrow \text{const}}}$$

minimize this for special case Lagrangian is (Maupertuis') principle of least action

= seek extremum trajectory in configuration space with:

- Energy E fixed
- time variable
- end points in configuration space fixed.