De Broglie's "Theorem of Phase Agreement"

Periodic phenomena fixed to moving particle ... frequency

$$\nu_1 = \frac{m_0 c^2}{h}\sqrt{1-\beta^2}$$

$$= \nu_0 \sqrt{1-\beta^2}$$

w.r.t. observer at rest

appears to this observer always in phase with

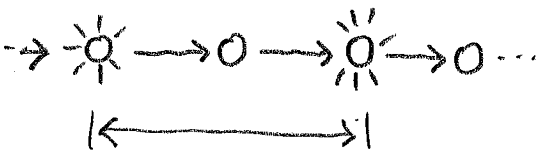
a wave of frequency

$$\nu = \frac{m_0 c^2}{h}\frac{1}{\sqrt{1-\beta^2}}$$

$$= \nu_0 / \sqrt{1-\beta^2}$$

propagating in the direction of motion of the moving particle at... $V = c/\beta = c^2/v$

$v \rightarrow \quad \beta = v/c$

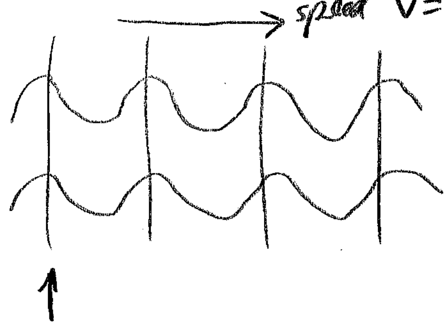


Frequency ν_1

Time $1/\nu_1$ in rest frame

$\nu_1 = \sqrt{1-\beta^2}\,\nu_0$

speed $V = c^2/v$

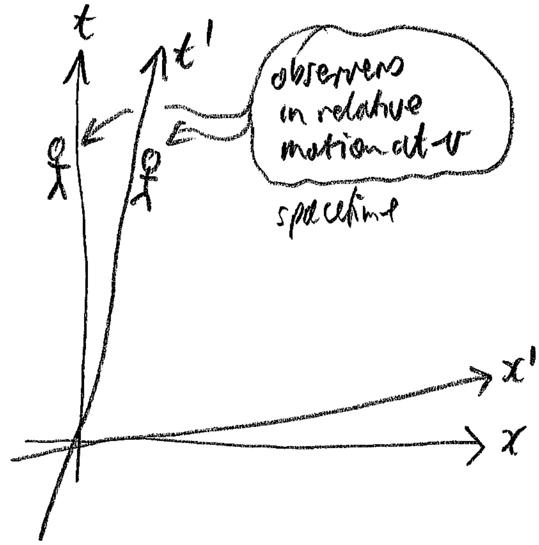


Frequency $\nu = \frac{\nu_0}{\sqrt{1-\beta^2}}$

ν_0 = proper frequency of object

$h\nu_0 = m_0 c^2$

Lorentz transformation

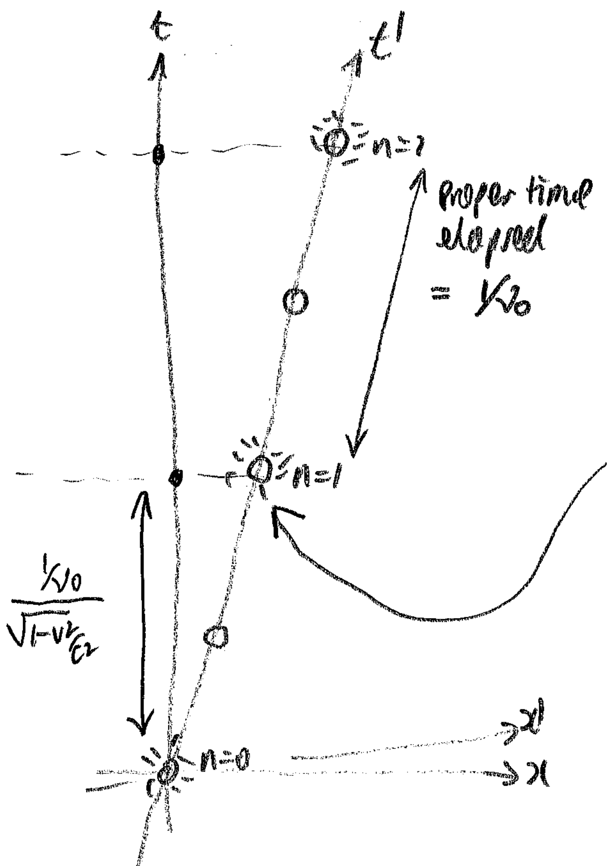


$$t' = \gamma\left(t - \tfrac{v}{c^2}x\right)$$
$$x' = \gamma\left(x - vt\right)$$
$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

$$t = \gamma\left(t' + \tfrac{v}{c^2}x'\right)$$
$$x = \gamma\left(x' + vt'\right)$$

Particle with proper frequency $\nu_0 = \frac{m_0 c^2}{h}$ at rest in primed frame

Particle moves at
$x = vt$
$t = \frac{1}{v}x$



$n=1$ flash happens at
$(x', t') = (0, 1/\nu_0)$

$\Downarrow$ transform to unprimed frame

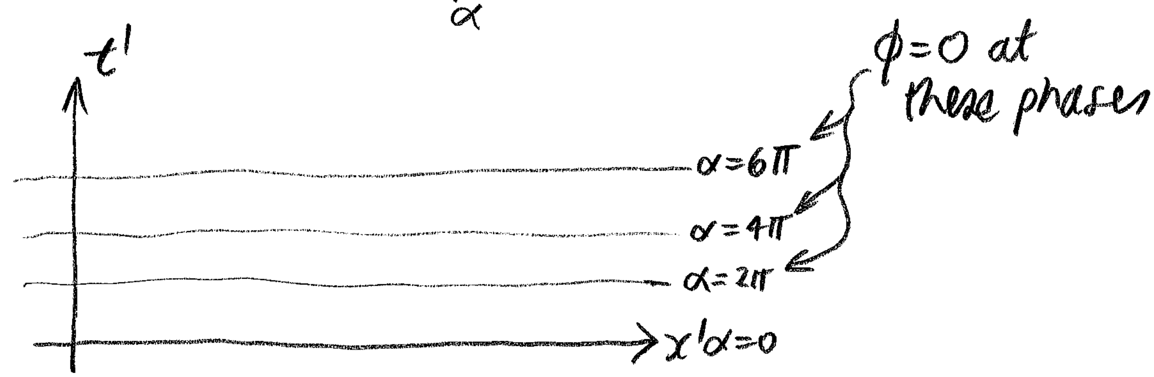
$$t = \gamma\left(t' + \tfrac{v}{c^2}x'\right) = \gamma t' = \frac{1/\nu_0}{\sqrt{1 - v^2/c^2}}$$

Hence frequency in unprimed frame is

$$\nu_1 = \nu_0\sqrt{1 - v^2/c^2}$$

Plane wave with proper frequency ν_0 for a co-moving observer

$$\phi(x',t') = \sin(\underbrace{2\pi\nu_0 t'}_{\alpha})$$



Transform to unprimed frame

$$\phi(x,t) = \sin(2\pi\nu_0 \overbrace{\gamma(t - \tfrac{v}{c^2}x)}^{t'})$$

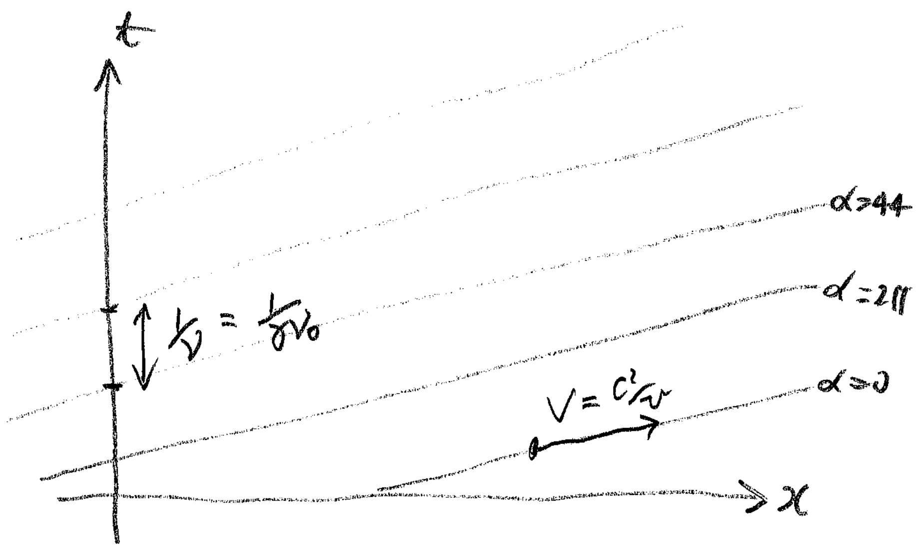
$$= \sin\Big(2\pi\big(\underbrace{\gamma\nu_0}_{\text{Frequency}} t - \underbrace{\frac{\gamma\nu_0 v}{c^2}}_{1/\text{wavelength}} x\big)\Big)$$

Frequency $\nu = \gamma\nu_0 = \nu_0/\sqrt{1 - v^2/c^2}$

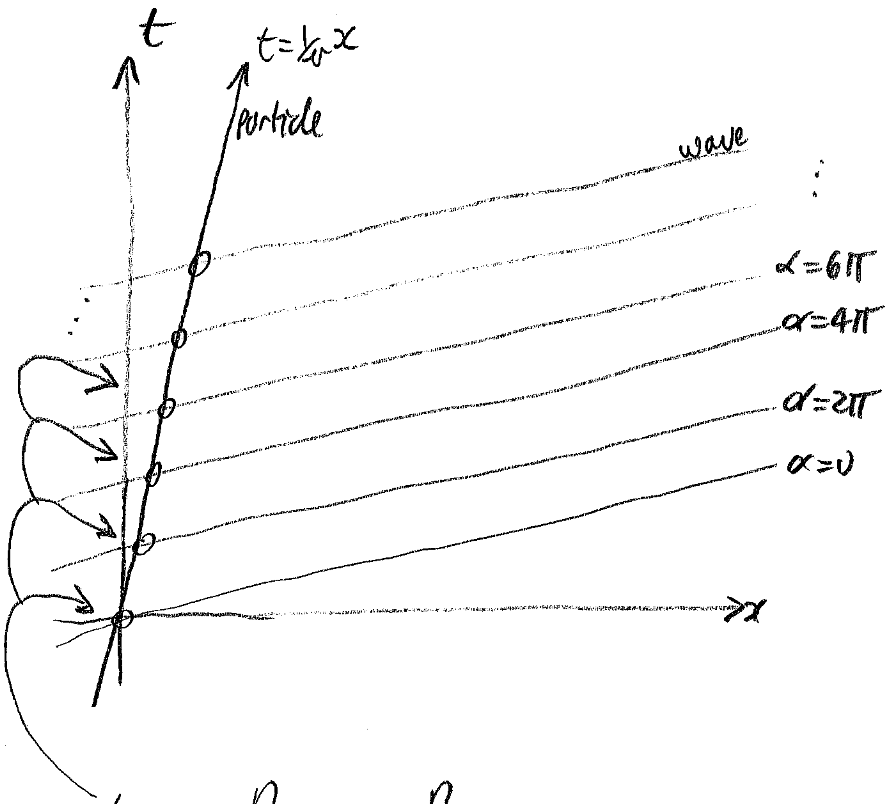
Phase velocity = velocity of a point of constant phase

Phase $= 0 = 2\pi(\gamma\nu_0 t - \gamma\frac{\nu_0 v}{c^2}x)$

$$\therefore \underbrace{x = \frac{\gamma\nu_0}{\gamma\nu_0 v/c^2}\, t = \frac{c^2}{v}x}_{\text{phase velocity is } V = c^2/v}$$



Superimpose particle and wave



Points of 360 phase when

$\alpha = 2\pi n \quad n = 0, 1, 2, \ldots$

i.e.

$2\pi n = 2\pi \nu_0 \gamma \left(t - \frac{v}{c^2} x\right)$

Intersect particle world line when $x = vt$

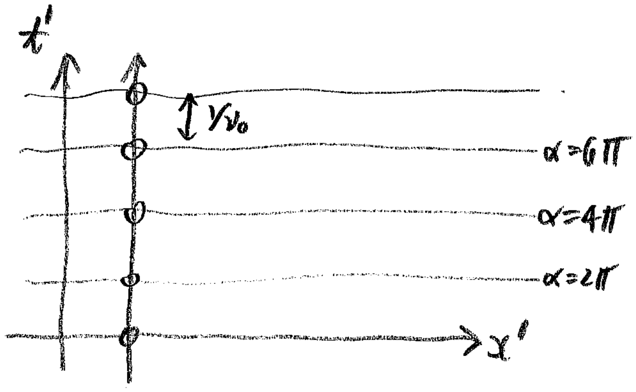
$$n = \nu_0 \gamma \left(t - \frac{v}{c^2} vt\right) = \nu_0 \sqrt{1 - v^2/c^2}\, t$$

$$\underbrace{t_n = \frac{n}{\nu_0 \sqrt{1 - v^2/c^2}} = \frac{n}{\nu_1}}$$

i.e. in unprimed frame, flashes separated by time $1/\nu_1$

Easy way to see result

① Particle & wave in particle rest frame



$$\phi = \sin\alpha = \sin(2\pi \nu_0 t')$$

② Transform to unprimed frame

De Broglie's:	Groupvelocity of the phase waves	is equal to	the velocity of the moving particle

Add two waves of adjacent frequency $\nu, \nu+\delta\nu$

Use identity

$$\sin(x)+\sin(x+\delta x)$$

$$= \sin(\bar{x}-\tfrac{\delta x}{2})+\sin(\bar{x}+\tfrac{\delta x}{2}) \qquad \bar{x}=x+\frac{\delta x}{2}$$

$$= \sin(\bar{x})\cos(-\tfrac{\delta x}{2})+\cos(\bar{x})\sin(-\tfrac{\delta x}{2})$$
$$+\sin(\bar{x})\cos(+\tfrac{\delta x}{2})+\cos(\bar{x})\sin(+\tfrac{\delta x}{2})$$

cancel $\sin(-\frac{\delta x}{2}) = -\sin(\frac{\delta x}{2})$

$$= 2\sin\bar{x}\cdot\cos\frac{\delta x}{2} \approx 2\sin x\cos\frac{\delta x}{2}$$

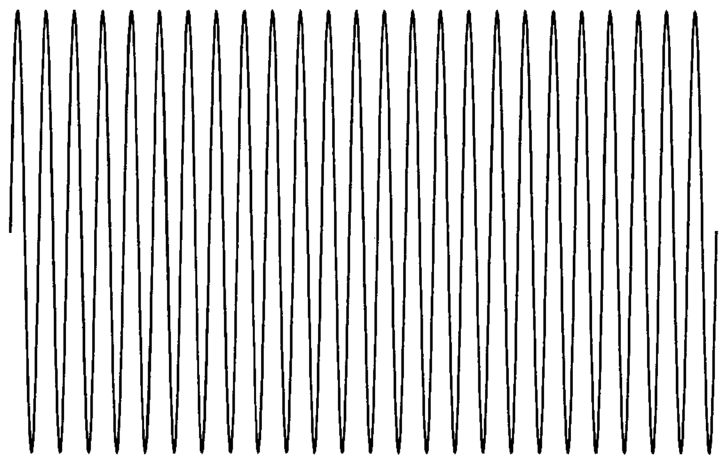
Apply to waveform $\sin 2\pi(\nu t - \frac{\nu}{V}x)$ to get "beats"

combined form is

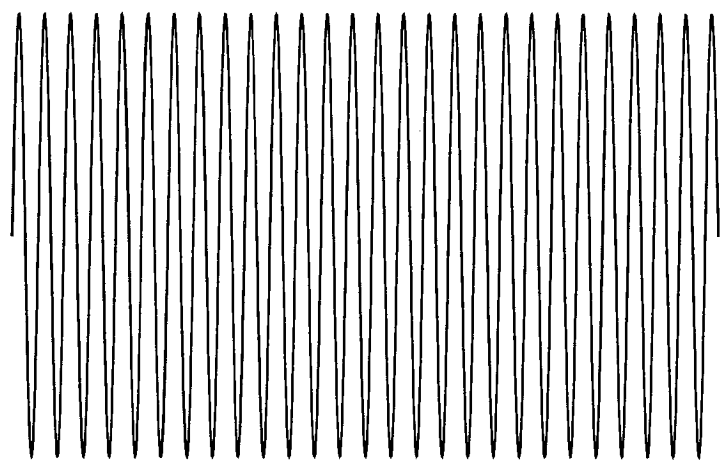
$$2\underbrace{\sin 2\pi\left(\nu t-\frac{\nu}{V}x\right)}_{\text{sine wave propagating with phase velocity } V=\frac{c^2}{v}}\cdot\underbrace{\cos 2\pi\left(\delta\nu t - x\,\delta(\nu/V)\right)}_{\text{modulating beat envelope propagating at } u \text{ where}}$$

$$\underbrace{\frac{1}{u}=\frac{\delta(\nu/V)}{\delta\nu}\rightarrow\frac{d}{d\nu}(\nu/V)}_{\text{group velocity}}$$

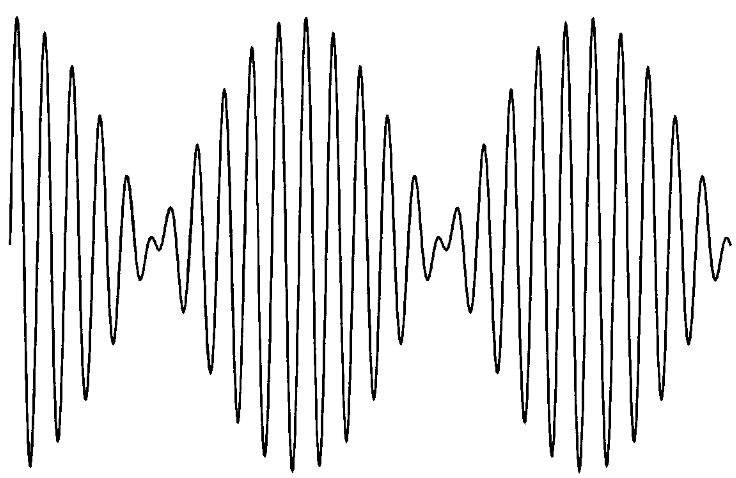
"Beats"



$\sin(x)$



$\sin(1.1x)$



$\sin(x)$
$+ \sin(1.1x)$

$\boxed{\text{compute group velocity}}$

$$\frac{1}{u} = \frac{d(\nu/V)}{d\nu}$$

Note functional dependencies!

$$\nu = \frac{1}{h}\frac{m_0 c^2}{\sqrt{1-v^2/c^2}} = \frac{\nu_0}{\sqrt{1-v^2/c^2}} \qquad V = \frac{c^2}{v} \qquad \frac{1}{V} = \frac{v}{c^2}$$

Reconfigure so v is the independent variable:

$$u = \frac{d\nu}{d(\nu/V)} = \frac{d\nu/dv}{\frac{d(\nu/V)}{dv}} = \frac{\nu_0 \frac{d}{dv}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)^{Ⓐ}}{\frac{\nu_0}{c^2}\frac{d}{dv}\left(\frac{v}{\sqrt{1-v^2/c^2}}\right)^{Ⓑ}}$$

$$= \frac{\nu_0 \frac{v/c^2}{(1-v^2/c^2)^{3/2}}}{\frac{\nu_0}{c^2}\frac{1}{(1-v^2/c^2)^{3/2}}} = v \quad ***$$

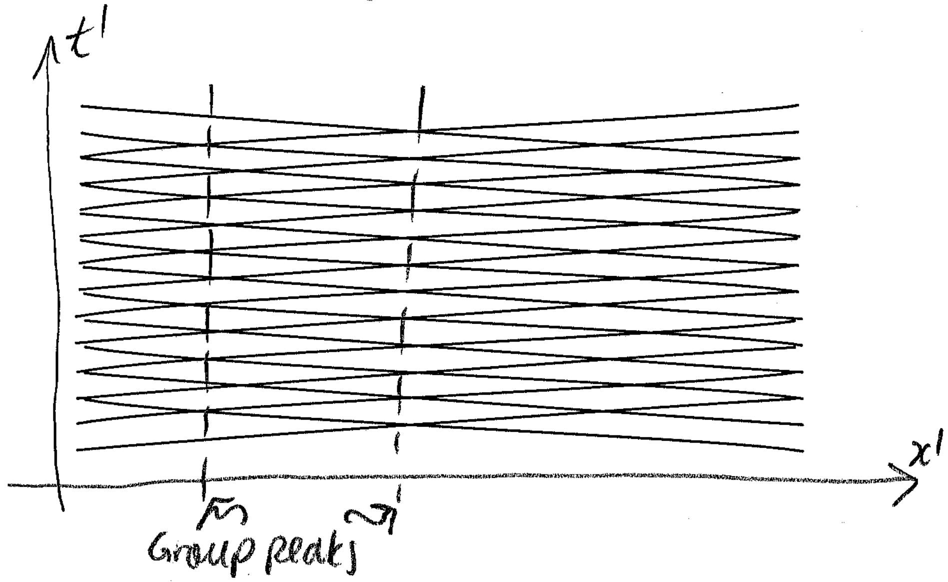
since Ⓐ $\frac{d}{dv}\frac{1}{(1-v^2/c^2)^{1/2}} = \frac{-\frac{1}{2}}{(1-v^2/c^2)^{3/2}} \cdot \frac{-2v}{c^2} = \frac{v/c^2}{(1-v^2/c^2)^{3/2}}$

Ⓑ $\frac{d}{dv}\frac{v}{(1-v^2/c^2)^{1/2}} = \frac{1}{(1-v^2/c^2)^{1/2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}} = \frac{(1-v^2/c^2)}{(1-v^2/c^2)^{3/2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}}$

$$= \frac{1}{(1-v^2/c^2)^{3/2}}$$

Group velocity / phase velocity relationship in a picture

Add two plane waves: one moves at $\frac{c^2}{\delta v} \approx +\infty$ one moves at $\frac{c^2}{-\delta v} \approx -\infty$ δv very small



By symmetry, interference pattern produces a wave at rest.

Transform to frame in which peaks move at v