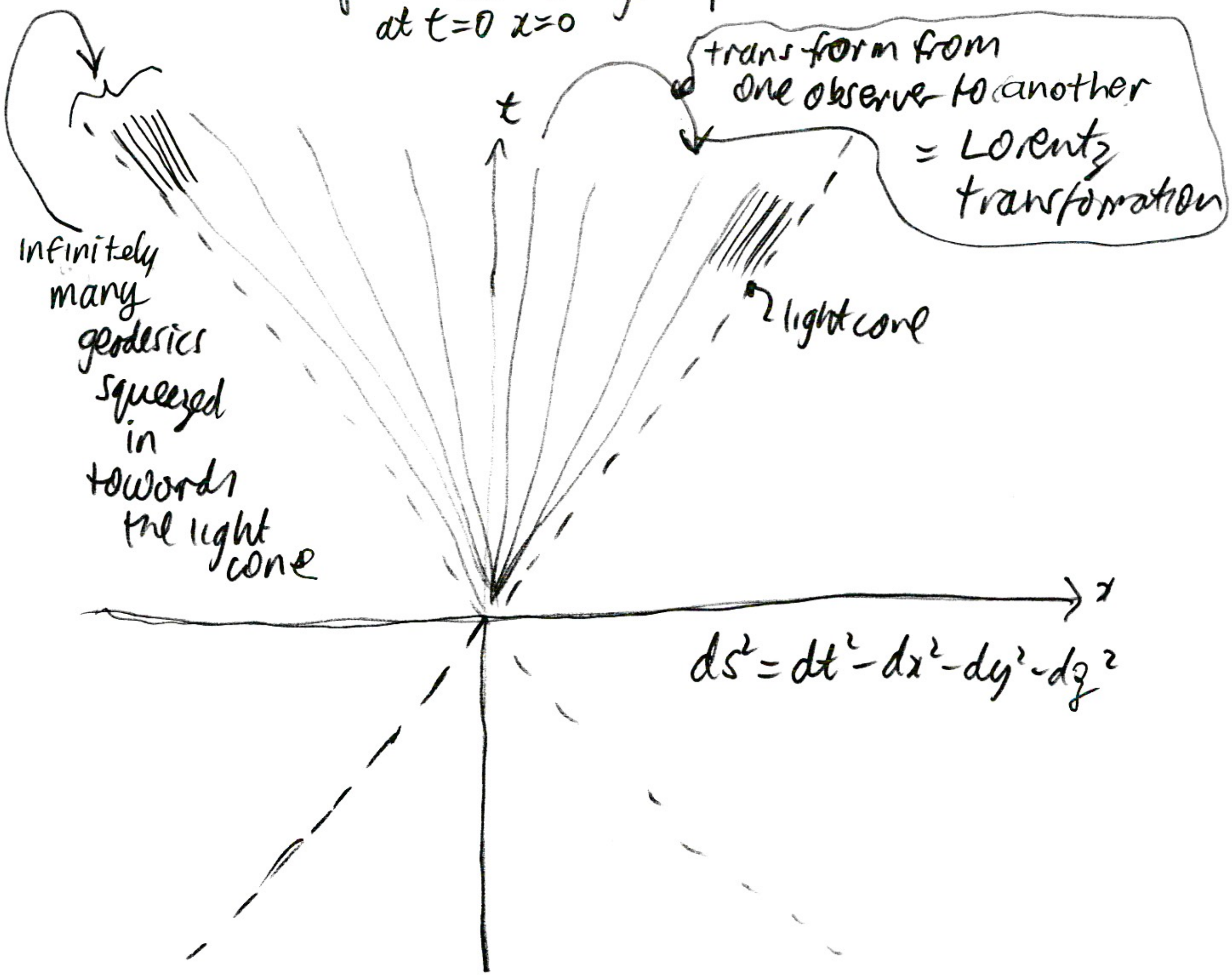
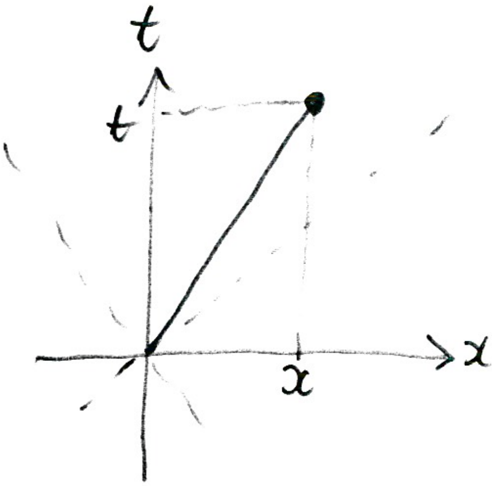
Milne's cosmology
Based on Bondi, cosmology, ch. XI

Basic structure: Minkowski space time.
Fundamental observers
= timelike geodesics emanating from an origin point at $t=0$ $x=0$



$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

Hubble law



Speed of galaxy

$$v = \frac{dx}{dt} = \frac{x}{t} = \underbrace{\left(\frac{1}{t}\right)}_{\text{Hubble constant} \,=\, 1/\text{age universe}} x$$

Recovered from Bondi p. 130 $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ are co-moving coordinates

$\bar{t}$: Proper time elapsed along fundamental observers' worldlines

$$t = \bar{t}\,\frac{1+\frac{1}{4}(\bar{x}^2+\bar{y}^2+\bar{z}^2)}{1-\frac{1}{4}(\bar{x}^2+\bar{y}^2+\bar{z}^2)}$$

$$x = \bar{t}\,\frac{\bar{x}}{1-\frac{1}{4}(\bar{x}^2+\bar{y}^2+\bar{z}^2)}$$

Eliminate $\bar{t}$

$$x = \underbrace{\frac{\bar{x}}{1+\frac{1}{4}(\bar{x}^2+\bar{y}^2+\bar{z}^2)}}_{\text{constant for each fundamental observer}}\; t$$

$$\boxed{\text{since } -1 < \frac{x}{t} < 1 \text{ we must have } -1 < \frac{\bar{x}}{1+\frac{1}{4}(\bar{x}^2+\bar{y}^2+\bar{z}^2)} < 1}$$

After some algebra, equivalent to $-2 < \bar{x} < 2$ if we set $\bar{y} = \bar{z} = 0$

Hyperbolic space – line element in co-moving coordinates

Bondi p. 130

Hence $\bar t$ is proper time along fundamental observers' world lines

$$ds^2 = d\bar t^2 - \bar t^2 \underbrace{\frac{d\bar x^2 + d\bar y^2 + d\bar z^2}{\left[1 - \frac{1}{4}(\bar x^2 + \bar y^2 + \bar z^2)\right]^2}}_{\text{line element of hyperbolic space}} = dt^2 - dx^2 - dy^2 - dz^2$$

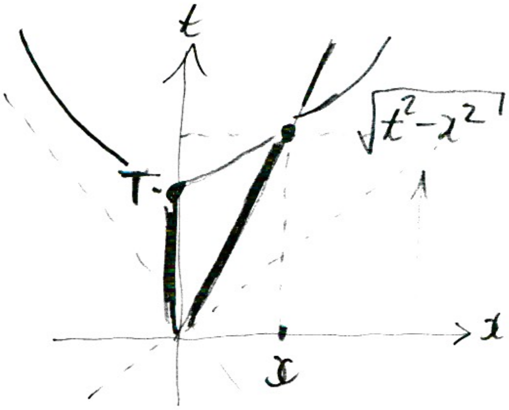
why?

$\Downarrow$

surface of constant $\bar t$ = surface of constant proper time elapsed along fundamental observers' world lines

$\Downarrow$

$$\overbrace{T}^{\text{proper time}} = \sqrt{t^2 - x^2}$$



$\boxed{T^2 = t^2 - x^2}$ Minkowski's original hyperbola

∴ Full 3-D surface is a hyperboloid of rotation

Has to be a hyperbola since the figure must be an invariant of the Lorentz transformation connecting fundamental observers' world lines