MATHEMATICAL RECREATIONS AND ESSAYS

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CHAPTER XI.

MISCELLANEOUS PROBLEMS.

I propose to discuss in this chapter the mathematical theory of a few common mathematical amusements and games. I might have dealt with them in the first four chapters, but, since most of them involve mixed geometry and algebra, it is rather more convenient to deal with them apart from the problems and puzzles which have been described already; the arrangement is, however, based on convenience rather than on any logical distinction.

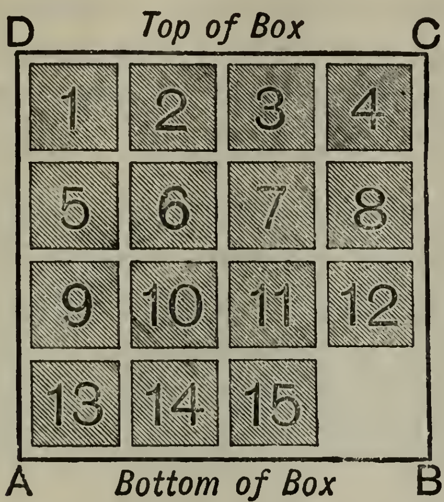
The majority of the questions here enumerated have no connection one with another, and I jot them down almost at random.

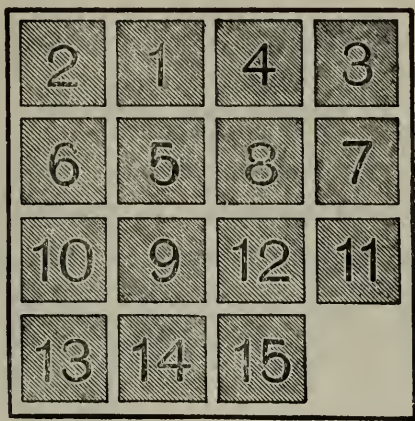
I shall discuss in succession the *Fifteen Puzzle*, the *Tower of Hanoï*, *Chinese Rings*, and some miscellaneous *Problems connected with a Pack of Cards*.

The Fifteen Puzzle*. Some years ago the so-called *Fifteen Puzzle* was on sale in all toy-shops. It consists of a shallow wooden box—one side being marked as the top—in the form of a square, and contains fifteen square blocks or counters numbered 1, 2, 3, ... up to 15. The box will hold just sixteen such counters, and, as it contains only fifteen, they can be moved about in the box relatively to one another. Initially they are put in the box in any order, but leaving the sixteenth

* There are two articles on the subject in the *American Journal of Mathematics*, 1879, vol. II, by Professors Woolsey Johnson and Storey; but the whole theory is deducible immediately from the proposition I give in the text.

cell or small square empty; the puzzle is to move them so that finally they occupy the position shown in the first of the annexed figures.





We may represent the various stages in the game by supposing that the blank space, occupying the sixteenth cell, is moved over the board, ending finally where it started.

The route pursued by the blank space may consist partly of tracks followed and again retraced, which have no effect on the arrangement, and partly of closed paths travelled round, which necessarily are cyclical permutations of an odd number of counters. No other motion is possible.

Now a cyclical permutation of n letters is equivalent to $n-1$ simple interchanges; accordingly an odd cyclical permutation is equivalent to an even number of simple interchanges. Hence, if we move the counters so as to bring the blank space back into the sixteenth cell, the new order must differ from the initial order by an even number of simple interchanges. If therefore the order we want to get can be obtained from this initial order only by an odd number of interchanges, the problem is incapable of solution; if it can be obtained by an even number, the problem is possible.

Thus the order in the second of the diagrams given above is deducible from that in the first diagram by six interchanges; namely, by interchanging the counters 1 and 2, 3 and 4, 5 and 6, 7 and 8, 9 and 10, 11 and 12. Hence the one can be deduced from the other by moving the counters about in the box.

If however in the second diagram the order of the last three counters had been 13, 15, 14, then it would have required seven interchanges of counters to bring them into the order given in the first diagram. Hence in this case the problem would be insoluble.

The easiest way of finding the number of simple interchanges necessary in order to obtain one given arrangement from another is to make the transformation by a series of cycles. For example, suppose that we take the counters in the box in any definite order, such as taking the successive rows from left to right, and suppose the original order and the final order to be respectively

	1,	13,	2,	3,	5,	7,	12,	8,	15,	6,	9,	4,	11,	10,	14,
and	11,	2,	3,	4,	5,	6,	7,	1,	9,	10,	13,	12,	8,	14,	15.

We can deduce the second order from the first by 12 simple interchanges. The simplest way of seeing this is to arrange the process in three separate cycles as follows:—

1, 11, 8;	13, 2, 3, 4, 12, 7, 6, 10, 14, 15, 9;	5.
11, 8, 1;	2, 3, 4, 12, 7, 6, 10, 14, 15, 9, 13;	5.

Thus, if in the first row of figures 11 is substituted for 1, then 8 for 11, then 1 for 8, we have made a cyclical interchange of 3 numbers, which is equivalent to 2 simple interchanges (namely, interchanging 1 and 11, and then 1 and 8). Thus the whole process is equivalent to one cyclical interchange of 3 numbers, another of 11 numbers, and another of 1 number. Hence it is equivalent to $(2 + 10 + 0)$ simple interchanges. This is an even number, and thus one of these orders can be deduced from the other by moving the counters about in the box.

It is obvious that, if the initial order is the same as the required order except that the last three counters are in the order 15, 14, 13, it would require one interchange to put them in the order 13, 14, 15; hence the problem is insoluble.

If however the box is turned through a right angle, so as to make AD the top, this rotation will be equivalent to 13 simple interchanges. For, if we keep the sixteenth square always blank, then such a rotation would change any order such as

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,
to 13, 9, 5, 1, 14, 10, 6, 2, 15, 11, 7, 3, 12, 8, 4,

which is equivalent to 13 simple interchanges. Hence it will change the arrangement from one where a solution is impossible to one where it is possible, and vice versa.

Again, even if the initial order is one which makes a solution impossible, yet if the first cell and not the last is left blank it will be possible to arrange the fifteen counters in their natural order. For, if we represent the blank cell by b, this will be equivalent to changing the order

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, b,
to b, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15:

this is a cyclical interchange of 16 things and therefore is equivalent to 15 simple interchanges. Hence it will change the arrangement from one where a solution is impossible to one where it is possible, and vice versa.

So, too, if it were permissible to turn the 6 and the 9 upside down, thus changing them to 9 and 6 respectively, this would be equivalent to one simple interchange, and therefore would change an arrangement where a solution is impossible to one where it is possible.

It is evident that the above principles are applicable equally to a rectangular box containing mn cells or spaces and $mn-1$ counters which are numbered. Of course m may be equal to n. If such a box is turned through a right angle, and m and n are both even, it will be equivalent to $mn-3$ simple interchanges—and thus will change an impossible position to a possible one, and vice versa—but unless both m and n are even the rotation is equivalent to only an even number of interchanges. Similarly, if either m or n is even, and it is impossible to solve the problem when the last cell is left blank, then it will be possible to solve it by leaving the first cell blank.

The problem may be made more difficult by limiting the possible movements by fixing bars inside the box which will prevent the movement of a counter transverse to their directions. We can conceive also of a similar cubical puzzle, but we could not work it practically except by sections.

THE TOWER OF HANOÏ. I may mention next the ingenious puzzle known as the *Tower of Hanoï*. It was brought out in 1883 by M. Claus (Lucas).

It consists of three pegs fastened to a stand, and of eight circular discs of wood or cardboard each of which has a hole in the middle through which a peg can be passed. These discs are of different radii, and initially they are placed all on one peg, so that the biggest is at the bottom, and the radii of the successive discs decrease as we ascend: thus the smallest disc is at the top. This arrangement is called the *Tower*. The problem is to shift the discs from one peg to another in such a way that a disc shall never rest on one smaller than itself, and finally to transfer the tower (*i.e.* all the discs in their proper order) from the peg on which they initially rested to one of the other pegs.

The method of effecting this is as follows. (i) If initially there are n discs on the peg A, the first operation is to transfer gradually the top $n-1$ discs from the peg A to the peg B, leaving the peg C vacant: suppose that this requires x separate transfers. (ii) Next, move the bottom disc to the peg C. (iii) Then, reversing the first process, transfer gradually the $n-1$ discs from B to C, which will necessitate x transfers. Hence, if it requires x transfers of simple discs to move a tower of $n-1$ discs, then it will require $2x+1$ separate transfers of single discs to move a tower of n discs. Now with 2 discs it requires 3 transfers, *i.e.* 2^2-1 transfers; hence with 3 discs the number of transfers required will be $2(2^2-1)+1$, that is, 2^3-1. Proceeding in this way we see that with a tower of n discs it will require 2^n-1 transfers of single discs to effect the complete transfer. Thus the eight discs of the puzzle will require 255 single transfers. It will be noticed that every alternate move