Contraction of the length of a curve under uniform contraction of the enclosing space

Consider a curve in a space with Cartesian coordinates (x,y), where the curve is represented by the functional dependence

$$y = y(x)$$

The length of the curve between limits $x = a$ and $x = b$ is

$$L(a,b) = \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

We form a new curve

$$y' = y'(x')$$

contracted vertically by the constant factor k and horizontally by the same factor k. The new variables y' and x' related to the old variables y and x according to:

$$y' = ky \text{ and } x' = kx$$

(Informally, the value of the new curve y' at a point x' reduced by a factor of k from x is equal to the value of the original function $y(x)$ at x, but now also reduced by a factor of k.) The new curve extends between the limits $x' = ka$ and $x' = kb$. We have for the length of the new curve:

$$L'(ka,kb) = \int_{x'=ka}^{kb} \sqrt{1 + \left(\frac{dy'}{dx'}\right)^2}\, dx'$$

Substituting unprimed variables for primed variables, this length becomes

$$L'(ka,kb) = \int_{kx=ka}^{kb} \sqrt{1 + \left(\frac{d(ky)}{d(kx)}\right)^2}\, d(kx) = k \int_{x=a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = kL(a,b)$$

That is, the length of the curve is also contracted by a factor k.

Inspection of the calculation shows that the result depends on both variables x and y being contracted by the same factor k. Otherwise, we have a failure of the cancellations of factors:

$$\left(\frac{dy'}{dx'}\right) = \left(\frac{d(ky)}{d(kx)}\right) = \left(\frac{dy}{dx}\right)$$