Non-Existence of Limit Set

Supplement to Badly Behaved Curves
John D. Norton

Consider the sets of real numbers for all $n \in \mathbb{N}$

$C_1 = (1/2, 1)$

$C_2 = (1/4, 1/2) \cup (3/4, 1)$

$C_3 = (1/8, 1/4) \cup (3/8, 1/2) \cup (5/8, 3/4) \cup (7/8, 1)$

…

$$C_n = \bigcup_{m=1}^{2^{n-1}} \left(\frac{2m-1}{2^n}, \frac{2m}{2^n} \right) \tag{1}$$

…

Here $(a,b) = \{x: a < x < b\}$, so that the interval is open. That is, it is the set of real numbers between a and b, *excluding* a and b. The result shown here is that there is no well-defined limit set formed by taking the limit at $n \to \infty$ of the sets C_n. The members of the limit set are defined by the condition

$$x \in C_{lim} \quad \text{iff} \quad \text{there is an } N \text{ such that } x \in C_n \text{ for all } n > N. \tag{2}$$

The non-existence of the set follows from two results:

(a) If $x = r/2^N$ for some N and natural number $r \leq N$, then $x \notin C_n$ for all $n > N$. It follows from (1) that $x \notin C_{lim}$.

(b) If $x \neq r/2^N$ for some N and natural number $r \leq N$, then x never satisfies condition (2). Rather for every x and n such that $x \in C_n$, there is an $n' > n$, such that $x \notin C_{n'}$; and for every x and n such that $x \notin C_n$, there is an $n' > n$, such that $x \in C_{n'}$.

The result (a) is compatible with the limit set existing but being empty. Result (b) is more troublesome since (2) does not enable to say whether the values of x to which it applies are in the set or not. Hence the set is not empty, but not well defined.

Proof of (a)

To see (a), consider a real x such that $x = r/2^N$ for some natural number N and natural number $r \leq N$. Then there is an m in the formula (1) such that $x = r/2^N = (2m+1)/2^N$ or $x = r/2^N =$

$2m/2^N$. That is, x is one of the extremal reals in the specification of the open sets of (1). Since the intervals in (1) are open, it follows that $x \notin C_N$.

It now also follows that $x \notin C_{N+1}$. For x will now be one of the extremal reals in the specification of the open sets of C_{N+1}. For $x = 2r/2^{N+1} = (4m+2)/2^{N+1}$ or $x = 2r/2^{N+1} = 4m/2^{N+1}$. Iterating, it follows that $x \notin C_{N+2}$, $x \notin C_{N+3}$, … and so on for all C_n with $n > N$.

Proof of (b)

If $x \neq r/2^N$ for some N and natural number $r \leq N$, then, for any n, x is not one of the extremal reals used to specify the open sets in C_n. To proceed, pick any $n > 1$. (The choice will not affect the result.) There must exist some value of m in (1) such that

either (i) $x \in \left(\frac{2m-1}{2^n}, \frac{2m}{2^n}\right)$ or (ii) $x \in \left(\frac{2m}{2^n}, \frac{2m+1}{2^n}\right)$.

In case (i), we have that $x \in C_n$. The quick way to see this is to note that the open sets included in C_n have the form $\left(\frac{\text{odd number}}{2^n}, \frac{\text{even number}}{2^n}\right)$. The sets excluded from C_n have the form $\left(\frac{\text{even number}}{2^n}, \frac{\text{odd number}}{2^n}\right)$. Since $x \neq (4m\text{-}1)/2^{n+1}$, we must have that

either (i.a) $x \in \left(\frac{4m-2}{2^{n+1}}, \frac{4m-1}{2^{n+1}}\right)$ or (i.b) $x \in \left(\frac{4m-1}{2^{n+1}}, \frac{4m}{2^{n+1}}\right)$.

In case (i.a), we have that $x \notin C_{n+1}$ since (i.a) has the form $\left(\frac{\text{even number}}{2^{n+1}}, \frac{\text{odd number}}{2^{n+1}}\right)$. If, however, we have case (i.b), then $x \in C_{n+1}$, since x lies in an interval of the form $\left(\frac{\text{odd number}}{2^n}, \frac{\text{even number}}{2^n}\right)$. In this case (i.b), we repeat the analysis and check whether $x \in C_{n+2}$; and so on for C_{n+3} etc. Eventually we must find a C_N with $N > n$ such that $x \notin C_N$. For otherwise, x can be brought arbitrarily close to a real number of the form (even number / 2^N) for some N. This can only be the case if x has the form (even number / 2^N) for some N. However, by supposition of case (i), x does not have this form. Hence in either case (i.a) or (i.b), we eventually find a value of $N > n$, such that $x \notin C_N$. That is, if $x \in C_n$, there exists $N > n$, such that $x \notin C_N$.

Case (ii) above is the case of x a member of an open set of the form $\left(\frac{\text{even number}}{2^n}, \frac{\text{odd number}}{2^n}\right)$, so that $x \notin C_n$. By reasoning analogous to that of case (i), we find that there exists $N > n$, such that $x \in C_N$.

The case of x = 1/3

This is a simple case of a number for which there is no definite limiting fact over its membership in the limit set. This failure of the limit fact arises because x = 1/3 alternatives in its membership of the sets C_n indefinitely according to:

$$1/3 \notin C_1,\ 1/3 \in C_2,\ 1/3 \notin C_3,\ 1/3 \in C_4,\ \ldots$$

That is, we have $1/3 \notin C_n$, when n is odd; and $1/3 \in C_n$, when n is even.

Approximations for 1/3

To arrive at these results, we need some approximation formulae for 1/3. We have that

$$1/3 = 1/4 + 1/16 + 1/64 + \ldots + 1/2^{2n} + \ldots$$

We can split this series into two terms,

$$1/3 = \textit{lower sum} + \textit{error}$$

where

$$\textit{lower sum} = 1/4 + 1/16 + 1/64 + \ldots + 1/2^{2n} = \frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right)$$

$$\textit{error} = 1/2^{2n+1} + 1/2^{2n+2} + \ldots = \frac{1}{3}\left(\frac{1}{2^{2n}}\right)$$

For the first approximation, we have that $0 < \textit{error} = \frac{1}{3}\left(\frac{1}{2^{2n}}\right) < \left(\frac{1}{2^{2n}}\right)$. It follows that

$$\frac{1}{3} \in \left(\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right), \frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right) + \frac{1}{2^{2n}}\right) \tag{3}$$

A tighter approximation arises from $0 < \textit{error} = \frac{1}{3}\left(\frac{1}{2^{2n}}\right) < \frac{1}{2}\left(\frac{1}{2^{2n}}\right)$. It follows that

$$\frac{1}{3} \in \left(\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right), \frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right) + \frac{1}{2}\frac{1}{2^{2n}}\right) \tag{4}$$

$1/3 \in C_n$, when n is even

This result follows from approximation (3). To use it, we need to show that

$$(2^{2n} - 1)/3 \text{ is an odd number}$$

To see this, sum the series

$$1 + 4 + 4^2 + \ldots + 4^{n-1} = \frac{2^{2n}-1}{2^2-1} = \frac{2^{2n}-1}{3}$$

The sum on the left is a sum of n-2 even numbers and 1. Hence it is odd, as must be the term of interest on the right. Applying this to approximation (3), we have that

$$\frac{1}{3} \in \left(\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right), \frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right) + \frac{1}{2^{2n}}\right) = \left(\frac{\text{odd number}}{2^{2n}}, \frac{\text{next even number}}{2^{2n}}\right)$$

Hence it follows that $1/3 \in C_{2n}$, or that $1/3 \in C_n$, when n is even.

$1/3 \notin C_n$, when n is odd

To see this, we use the approximation (4). If the fractions delimiting the open set are multiplied by 2/2, we recover:

$$\frac{1}{3} \in \left(\frac{2}{3}\left(\frac{2^{2n}-1}{2^{2n+1}}\right), \frac{2}{3}\left(\frac{2^{2n}-1}{2^{2n+1}}\right) + \frac{1}{2^{2n}}\right)$$

We know from earlier that $(1/3)(2^{2n} - 1)$ is an odd number. Hence $(2/3)(2^{2n} - 1)$ is an even number. Thus the approximation becomes

$$\frac{1}{3} \in \left(\frac{\text{even number}}{2^{2n+1}}, \frac{\text{next odd number}}{2^{2n+1}}\right)$$

These open intervals are not subsets of C_{2n+1}. It follows that $1/3 \notin C_{2n+1}$, or that $1/3 \in C_n$, when n is odd.