JACOB BERNOULLI

THE ART OF CONJECTURING

together with

LETTER TO A FRIEND ON SETS IN COURT TENNIS

Translated with an introduction and notes by

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THE ART OF CONJECTURING
PART FOUR

Teaching

The Use and Application of the Preceding Doctrine in Civil, Moral, and Economic Matters

Chapter I. *Some preliminaries on the certainty, probability, necessity, and contingency of things*

The *certainty* of anything is considered either *objectively* and in itself or *subjectively* and in relation to us. Objectively, certainty means nothing else than the truth of the present or future existence of the thing. Subjectively, certainty is the measure of our knowledge concerning this truth.

In themselves and objectively, all things under the sun, which are, were, or will be, always have the highest certainty. This is evident concerning past and present things, since, by the very fact that they are or were, these things cannot not exist or not have existed. Nor should there be any doubt about future things, which in like manner, even if not by the necessity of some inevitable fate, **[211]** nevertheless by divine foreknowledge and predetermination, cannot not be in the future. Unless, indeed, whatever will be will occur with certainty, it is not apparent how the praise of the highest Creator's omniscience and omnipotence can prevail. Others may dispute how this certainty of future occurrences may coexist with the contingency and freedom of secondary causes; we do not wish to deal with matters extraneous to our goal.

Seen in relation to us, the certainty of things is not the same for all things, but varies in many ways, increasing and decreasing. Those things concerning the existence or future occurrence of which we can have no doubt—whether because of revelation, reason, sense, experience, αὐτοψίᾳ [autopsy, i.e., eyewitness], or other reasons—enjoy the highest, and absolute, certainty. All other things receive a less perfect measure of certainty in our minds, greater or less in proportion as there are more or fewer probabilities that persuade us that the thing is, will be, or was.

Probability, indeed, is degree of certainty, and differs from the latter as a part differs from the whole. Truly, if complete and absolute certainty, which we represent by the letter *a* or by 1, is supposed, for the sake of argument, to be composed of five parts or probabilities, of which three argue for the existence or future existence of some outcome and the others argue against it, then that outcome will be said to have $3a/5$ or $3/5$ of certainty.

One thing therefore is called *more probable* than another if it has a larger part of certainty, even though in ordinary speech a thing is called probable only if its probability notably exceeds one-half of certainty. I say *notably,* for what equals approximately half of certainty is called *doubtful* or undecided. Thus that which has $1/5$ of certainty is more probable than that which has $1/10$, even though neither one is positively probable.

Something is *possible* if it has even a very small part of certainty, impossible if it has none or infinitely little. Thus something that has $1/20$ or $1/30$ of certainty is possible.

Something is *morally certain* if its probability comes so close to complete certainty that the difference cannot be perceived. By contrast, something is *morally impossible* if it has only as much certainty as the amount by which moral certainty falls short of complete certainty. Thus if we take something that possesses $999/1000$ of certainty to be morally certain, **[212]** then something that has only $1/1000$ of certainty will be morally impossible.

Something is *necessary* if it cannot not exist, now, in the future, or in the past. This necessity may be physical, hypothetical, or contractual. It is *physically necessary* that fire burn, that a triangle have three angles equal to two right angles, and that a full moon occurring when the moon is at a node be eclipsed. It is *hypothetically necessary* that something, while it exists or has existed, or while it is assumed to exist or have existed, cannot not exist or not have existed. It is necessary in this sense that Peter, whom I know and posit to be writing, is writing. Finally, there is the *contractual* or *institutional necessity* by which a gambler who has thrown a six is said to win necessarily if the players have agreed beforehand that a throw of a six wins.

A thing that can now, in the future, or in the past *not* exist is *contingent* (either *free* depending on the will of a rational creature, or *fortuitous* and *haphazard* [*casuale*] depending on accident or fortune). This should be understood with reference to a remote rather than proximate power; nor does contingency always exclude all necessity even with respect to secondary causes. Let me clarify this by examples. It is most certain, given the position, velocity, and distance of a die from the gaming table at the moment when it leaves the hand of the thrower, that the die cannot fall other than the way it actually does fall. Likewise, given the present condition of the atmosphere, given the mass, position, motion, direction, and velocity of the winds, vapors, and clouds, and given the laws of the mechanism according to which all these things act on each other, tomorrow's weather cannot be other than what in fact it will be. Indeed, these effects follow from their own proximate causes no less necessarily than the phenomena of eclipses follow from the motion of the heavenly bodies. Yet it is customary to count only the eclipses as necessary and to count the fall of the die and future weather as contingent. The only reason for this is that those things which, to determine the subsequent effects, are supposed as given [*data*], and which indeed are given in nature, are not yet sufficiently known to us. And even if they were, the study of geometry and physics has not been sufficiently perfected to enable us to calculate from these givens [*ex datis*] their effects, in the way in which eclipses can be computed and predicted once the principles of astronomy are known. Before astronomy was brought to this degree of perfection, eclipses themselves, no less than these other two phenomena, had to be counted among future contingencies. It follows, therefore, that something can be seen as contingent by one person at one time which may be necessary to another person **[213]** (or even the same person) at another time, after its causes have become known. So contingency also mainly has reference to our knowledge, insofar as we see no contradiction in something not existing in the present or future, even if, here and now, by the force of a proximate cause unknown to us, it may necessarily exist or be produced.

We speak of *good fortune* (*un bonheur* in French, *ein Glück* in German) and *bad fortune* (*un malheur* in French, *ein Unglück* in German) when a good or bad thing happens to us not just in any way, but when it more probably, or at least equally probably, might not have happened. Accordingly, fortune is better (or worse) in proportion as it is less probable that this good (or bad) thing should have happened. Thus a person who finds a treasure by digging in the ground is remarkably fortunate, because this happens not once in a thousand times. If twenty deserters, one of whom is to be hung as an example to the others, compete for their lives in a game of chance, then the nineteen who are treated more kindly by lot are not properly said to be fortunate, but the twentieth, to whose lot the bad luck falls, is said to be most unfortunate. And your friend who emerges unharmed from a battle in which few of the combatants were killed should not be called fortunate, unless perhaps you think this should be said on account of the preeminence of the good attached to the preservation of life.

Chapter II. *On knowledge and conjecture. On the art of conjecturing. On the arguments for conjectures. Some pertinent general axioms.*

We are said to *know* or *understand* those things that are certain and beyond doubt, but only to *conjecture* or have *opinions* about all other things.

To *conjecture* about something is to measure its probability. Therefore we define the *art of conjecture*, or *stochastics*, as the art of measuring the probabilities of things as exactly as possible, to the end that, in our judgments and actions, we may always choose or follow that which has been found to be better, more satisfactory, safer, or more carefully considered. On this alone turns all the wisdom of the philosopher and all the practical judgment of the statesman. **[214]**

Probabilities are assessed according to the *number* together with the *weight* of the *arguments* that in any way prove or indicate that something is, will be, or has been. By *weight* I mean probative force.

These *arguments* are either *internal* or *external.* Internal or, as they are more commonly called, technical[1] arguments are taken from the topics—cause, effect, subject, associated circumstances, sign, or anything else that seems to have a connection with the thing to be proved. External and nontechnical arguments appeal to human authority and testimony. Here is an example. Titius is found slain on the road. Maevius is accused of having committed the murder. The arguments for the accusation are: 1. It is established that Maevius hated Titius. (This is an argument from *cause,* for hatred could have driven him to kill Titius.) 2. When Maevius was questioned, he turned pale and answered timidly. (This is an argument from *effect,* for his pallor and fear may have resulted from his consciousness of having committed the crime.) 3. A sword stained by blood was found in Maevius's house. (This is a *sign.*) 4. On the same day on which Titius was slain on the road, Maevius passed by that way. (This is a *circumstance* of place and time.) 5. Finally, Caius testifies that on the day before the murder was committed disputes had occurred between Titius and Maevius. (This is *testimony.*)

But before pursuing our project of showing how these arguments for conjectures are appropriately used to measure probabilities, I would first like to set down some general rules or axioms, which simple reason commonly suggests to a person of sound mind, and which the more prudent constantly observe in civil life.

1. *There is no place for conjectures in matters in which one may reach complete certainty.* Thus it would be pointless for an astronomer to want to conjecture about whether a particular full moon will be eclipsed or not from the fact that two or three are eclipsed every year, since he can find the truth of the matter by an infallible calculation. Similarly, if a thief responds under questioning that he sold a stolen item to Sempronius, and Sempronius is present, then it would be foolish for a judge to want to conjecture about the probability of the assertion from the face and tone of the speaker, or from the quality of the stolen item, or from other circumstances of the theft, since he can find out everything easily and certainly from Sempronius.

1. Latin: *artificialis,* literally made by art or technique.

2. *It is not sufficient to weigh one or another argument. Instead we must bring together* **[215]** *all arguments that we can come to know and that seem in any way to work toward a proof of the thing.* For example, three ships set sail from port. After some time it is reported that one of them has perished by shipwreck. Which do we conjecture it to be? If I considered only the number of ships, I might conclude that misfortune might equally well befall any of them. But since I remember that one of them was more eaten away by decay and age than the others, that it was badly equipped with sails and sail-yards, and also that it was commanded by a new and inexperienced skipper, I judge that it is surely more probable that this one perished than the others.

3. *We should pay attention not only to those arguments that serve to prove a thing, but also to all those that can be adduced for the contrary, so that, when both groups have been properly weighed, it may be established which arguments preponderate.* Of a friend who has been absent from the country for a very long time it is asked whether he can be declared dead. The following arguments support the affirmative. Despite every care being taken, nothing has been heard from him for a whole twenty years. Travelers to foreign parts are exposed to many dangers to life from which those who remain at home are exempt: perhaps, then, he died at sea, perhaps he was killed on the road, perhaps in battle, perhaps he died from disease or some fall in a place where no one knew him. If he were still alive, he would now be of an age that few reach even at home. Even if he lived at the ends of India, he would have written, because he knew that he could expect an inheritance at home. There are also other arguments. Nevertheless, one should not be satisfied with these arguments. Instead one should also oppose to them the following arguments, which support the negative. It is well known that the man was careless, that he disliked writing, that he slighted his friends. Perhaps he was taken captive by barbarians so that he could not write. Perhaps he did sometimes write from India, but the letters were lost either through the bearers' carelessness or through shipwreck. Finally, we know that many who have been away longer have returned uninjured in the end.

4. *Remote and universal arguments are sufficient for making judgments about universals, but when we make conjectures about individuals, we also need, if they are at all available, arguments that are closer and more particular to those individuals.* Thus when we are asked in the abstract how much more probable it is that a young man of twenty will outlive an old man of sixty rather than vice versa, there is nothing we can consider besides the difference in age. But when we discuss specific individuals, say the young man Peter and the old man Paul, we need to pay attention also to their particular constitutions and to the care **[216]** that each one takes of his health. For if Peter were sickly, if he indulged his passions, or if

he lived intemperately, then Paul, even though he is more advanced in age, might, with excellent reason, hope to live longer.

5. *In matters that are uncertain and open to doubt, we should suspend our actions until we learn more. But if the occasion for action brooks no delay, then between two actions we should always choose the one that seems more appropriate, safer, more carefully considered, or more probable, even if neither action is such in a positive sense.* Thus when a fire has broken out and you cannot escape unless you jump either from the highest roof or from some lower story, it will be better to choose the lower story because it is safer, even though neither choice is safe in an absolute sense and neither can be made without danger of injury.

6. *What may help in some case and can harm in none is to be preferred to that which in no case either helps or harms.* Along this same line is our common saying, "it may not help, but it will do no harm."[2] This follows from the preceding, for something that can benefit is better, safer, and more desirable, other things being equal, than something that cannot.

7. *We should not judge the value of human actions by their results,* since sometimes the most foolish actions enjoy the best success, while the most prudent actions have the worst result. On this the Poet: *I wish no success to him who supposes that deeds should be evaluated by their results.*[3] A person who undertakes to get three sixes the first time he rolls three dice is judged to have acted foolishly, even if he should happen to win. This should be said against the perverse popular judgment that whoever is more fortunate is more excellent. Indeed, a happy and prosperous person's crime is often called a virtue. Owen wrote elegantly against this:

> *Epigrams, Single Book, Section 216*
> Because what was badly advised fell out happily, Ancus is declared wise, who just now was foolish;
> Because of what was prudently prepared for, if it turns out badly, Cato himself, in popular opinion, will be foolish.[4]

8. *In our judgments we should be careful not to attribute more weight to things than they have. Nor should we consider something that is more probable than its alter-*

2. German: *Hilfft es nicht, so schadt es nicht.*

3. Latin: *Careat successibus, opto, quisquis ab eventu facta notanda putat.* This line appears in Ovid, *Heroides,* II, l. 87.

4. Latin: *Quod male consultum cecidit feliciter, Ancus arguitur sapiens, qui modo stultus erat; / Quod prudenter erat provisum, si male vortat, Ipse Cato populo judice stultus erit.* The author is John Owen or Audoenus (1616–83) of Wales. See J. R. C. Martyn, *Audoeni Epigrammatum,* vol. 1: Libri I–III, vol. 2: Libri IV–X (Leiden: Brill, 1976, 1978), and Jozef Ijsewijn, *Companion to Neo-Latin Studies* (Amsterdam: North-Holland, 1977), p. 121.

natives to be absolutely certain, or force it on others. For it is necessary that the confidence we ascribe to any particular thing be proportioned **[217]** to the degree of certainty the thing has and also that it be diminished in proportion as the probability of the thing is diminished. As we commonly say, "everything must be taken for what it is worth."[5]

9. *Because, however, it is rarely possible to obtain certainty that is complete in every respect, necessity and use ordain that what is only morally certain be taken as absolutely certain.* It would be useful, accordingly, if definite limits for moral certainty were established by the authority of the magistracy. For instance, it might be determined whether $\frac{99}{100}$ of certainty suffices or whether $\frac{999}{1000}$ is required. Then a judge would not be able to favor one side, but would have a reference point to keep constantly in mind in pronouncing a judgment.

Anyone, from daily life, could formulate for himself many more axioms of this kind, all of which we might have difficulty remembering outside the particular situation.

Chapter III. *Various kinds of arguments and how to assess their weights for computing probabilities of things.*

Examination of the various arguments from which opinions or conjectures are formed reveals a threefold distinction among them:

- some arguments *exist necessarily and indicate contingently,*
- some *exist contingently and indicate necessarily,*
- some *both exist contingently and indicate contingently.*

This distinction can be explained by examples. My brother has not written me for a long time; I am not sure whether his laziness or business is to blame, and I fear he may even have died. There are three arguments for his not having written: *laziness, death,* and *business.* The first of these exists necessarily (by a hypothetical necessity; I take it as known that my brother is lazy), but indicates contingently, for it could be that his laziness did not prevent him from writing. The second argument exists contingently (for my brother could still be alive) **[218]** but indicates necessarily, for a dead man cannot write. The third argument both exists contingently and indicates contingently, for he might or might not have any business, and, if he does, it might or might not be enough to prevent him from writing. Another example: according to the rules of a game, a dice player

5. German: *Man muß ein jedes in seinem Werth und Unwerth beruhen lassen.*

wins if he throws a seven with two dice. I want to conjecture how much hope he has of winning. Here the argument for his winning is a throw of seven, which indicates necessarily (the necessity established, to be sure, by the contract entered into by the players) but exists only contingently, since points other than seven may fall.

Besides this distinction among arguments, we can observe another: some arguments are *pure,* others are *mixed. Pure* arguments prove a thing in some cases in such a way that they prove nothing positively in other cases. *Mixed* arguments prove a thing in some cases in such a way that they prove the contrary in the other cases. Example: suppose someone in a milling crowd is stabbed with a sword and it is established by the testimony of reliable witnesses looking on from a distance that the perpetrator of the crime had on a black cloak; suppose further that among the crowd Gracchus along with three others is found wearing a cloak of this color. This cloak will be an argument for murder committed by Gracchus, but a mixed argument, since in one case it proves his guilt but in three cases his innocence, depending on whether he or one of the other three is the perpetrator; for one of the latter could not be the perpetrator without Gracchus by that very fact being innocent. If, however, in a subsequent interrogation Gracchus turns pale, his pallor is a pure argument: it proves Gracchus's guilt if it results from a guilty conscience; but it does not, on the other hand, prove his innocence if it has another origin. For it is possible that Gracchus could turn pale for another reason and still be the murderer.

It is clear from the foregoing that any argument's power of proof depends upon the number of cases in which the argument can exist or not exist, indicate or not indicate, or even indicate the contrary. So the degree of certainty, or probability, that the argument generates can be found from these cases by the doctrine of Part I, just as the lots of dice players are found in games of chance. In order to show this, let b represent the number of cases in which it may happen that some argument **[219]** exists, c the number in which it may happen that the argument does not exist, and $a = b + c$ the number of both [types of cases]. Similarly, let β represent the number of cases in which it may happen that an argument indicates, γ the number in which the argument does not indicate or indicates the contrary, and $\alpha + \beta = \gamma$ the number of both. I assume that all cases are equally possible, or can happen with equal ease. Otherwise a correction must be made. For any case that happens more easily than the others as many more cases must be counted as it more easily happens. For example, in place of a case three times as easy I count three cases each of which may happen as easily as the rest.

1. First, then, consider an argument that *exists contingently and indicates necessarily.* In our notation there are b cases in which it can exist and indicate the

thing (or 1), and c cases in which it can not exist and hence can indicate nothing. By Corollary 1 of Proposition III of Part I this is worth $[(b \cdot 1) + (c \cdot 0)]/a = b/a$ so that such an argument proves b/a of the thing, or of its certainty.

2. Next let the argument *exist necessarily and indicate contingently.* By hypothesis there will be β cases in which it can happen that it indicates the thing and γ cases in which it does not indicate anything or indicates the contrary. This now makes the argument's force for proving the thing $[(\beta \cdot 1) + (\gamma \cdot 0)]/\alpha = \beta/\alpha$. Therefore, an argument of this kind proves β/α of certainty of the thing, and also, if the argument is mixed, it proves (as is clear in the same way) $[(\gamma \cdot 1) + (\beta \cdot 0)]/\alpha = \gamma/\alpha$ of certainty of the contrary.

3. If an argument *exists contingently and indicates contingently,* I assume first that it exists, in which case it can be shown, as above, to prove β/α of the thing, and in addition, if it is mixed, γ/α of the contrary. Then since there are b cases in which it exists, and c cases in which it does not exist and so can prove nothing, the argument will be worth $[(b \cdot \beta/\alpha) + (c \cdot 0)]/a = b\beta/a\alpha$ for proving the thing, and if it is mixed it will be worth $[(b \cdot \gamma/\alpha) + (c \cdot 0)]/a = b\gamma/a\alpha$ for proving the contrary.[6] **[220]**

4. Again, suppose several arguments are assembled for the proof of the same thing, and denoted as follows:

Numbers of cases	*Numbers of arguments:* *First*	*Second*	*Third*	*Fourth*	*Fifth*	*etc.*
Total	a	d	g	p	s	etc.
Proving	b	e	h	q	t	etc.
Not-proving or proving contrary	c	f	i	r	u	etc.

Then the force of proof resulting from the concurrence of all the arguments is estimated as follows. First let all the arguments be *pure.* Then, as we have seen, the weight of the first argument considered alone will be $b/a = (a - c)/a$. (This stands for β/α if the argument indicates contingently, or for $b\beta/a\alpha$ if it also exists contingently.) Now consider another argument which in e or $d - f$ cases proves the thing (or 1), and in f cases proves nothing, so that the weight of the first argument alone, which has been shown to be $(a - c)/a$, remains effective; the weight from both arguments together will be

6. In these mathematical expressions, the smaller fractions in the numerators are actually printed as the ratios $\beta : \alpha$ and $\gamma : \alpha$.

$$\frac{(d-f)1+f\left(\frac{a-c}{a}\right)}{d}=\frac{ad-cf}{ad}=1-\frac{cf}{ad}$$

of the thing. Let a third argument be added; there will be h or $g-i$ cases that prove the thing, and i cases in which the argument is null and only the two earlier arguments retain their power of proof by themselves, $(ad-cf)/ad$, whence the force of all three is judged to be

$$\frac{[(g-i)1]+\left[i\frac{(ad-cf)}{ad}\right]}{g}=\frac{adg-cfi}{adg}=1-\frac{cfi}{adg}.$$

And so on successively if there are further arguments at hand. From this it is clear that all the arguments taken together lead to a probability that falls short of absolute certainty of the thing, or unity, by that part of unity that is obtained by dividing the product of the nonproving cases by the product of all the cases in all the arguments.

5. Next let all the arguments be *mixed.* Since the number of proving cases in the first argument is b, in the second e, and the third h, etc., and the number proving the contrary, c, f, i, etc., the probability of the thing is to the probability of the contrary as b is to c on the strength of the first argument alone, as e is to f on the strength of the second alone, and as h is to i on the strength of the third alone, etc. Hence it is sufficiently evident that the total force of proof resulting from the concurrence of all the arguments may be **[221]** composed of the forces of all the arguments taken singly, that is, that the probability of the thing to the probability of its contrary is in the ratio of beh etc. to cfi etc. Hence the absolute probability of the thing is $beh/(beh+cfi)$, and the absolute probability of the contrary is $cfi/(beh+cfi)$.

6. On the other hand, let some of the arguments be *pure* (say the first three) and some *mixed* (say the two others). Consider first the pure ones alone, which by section 4 prove $(adg-cfi)/adg$ of the certainty of the thing, falling short of unity by cfi/adg. There are, as it were, $adg-cfi$ cases in which these three arguments together prove the thing, or unity, and cfi cases in which they prove nothing and consequently give the mixed arguments alone the opportunity to prove something. But by section 5 above these two mixed arguments prove $qt/(qt+ru)$ of the thing and $ru/(qt+ru)$ of the contrary. So the probability of the thing resulting from all the arguments is

$$\frac{(adg-cfi)1+cfi\left(\frac{qt}{qt+ru}\right)}{adg}=\frac{adgqt+adgru-cfiru}{adgqt+adgru}=1-\frac{cfiru}{adg(qt+ru)},$$

which falls short of complete certainty or unity by the product of the part cfi/adg (the deficit from unity of the probability of the thing resulting from the pure arguments alone according to Rule 4) times $ru/(qt + ru)$, the absolute probability of the contrary computed from the mixed arguments by Rule 5 above.

7. Now if, besides the arguments that tend to prove a thing, other pure arguments for the contrary arise, then both categories of arguments must be weighed separately according to the preceding rules so as to establish the ratio that holds between the probability of the thing and the probability of the contrary. Here it should be noted that, if the arguments adduced on each side are strong enough, it may happen that the absolute probability of each side significantly exceeds half of certainty, that is, that both of the contraries are rendered probable, though relatively speaking one is less probable than the other. Thus it can happen that one thing has ⅔ of certainty, while its contrary has ¾; in this way both contraries will be probable, yet the first less probable than its contrary, in the ratio ⅔ to ¾, or 8 to 9.

I cannot conceal here that I foresee many problems in particular applications of these [222] rules that could cause frequent delusions unless one proceeds cautiously in discerning arguments. For sometimes arguments can seem distinct that in fact are one and the same. Or, vice versa, those that are distinct can seem to be one. Sometimes what is posited in one argument plainly overturns a contrary argument.

In illustration of this, I will give an example or two. Suppose that in the above example involving Gracchus the credible witnesses who saw the crowd observed also that the murderer had red hair. Gracchus along with two others was seen to have red hair, but neither of the other two was wearing a black cloak. Here a person would certainly be reasoning ineptly if, from the evidence that besides Gracchus three people wore black cloaks and that besides him two had red hair, he wanted to conclude that the probability of Gracchus's guilt to the probability of his innocence, by section 5, is in the ratio compounded from one-third and one-half, that is, in the ratio of one-sixth, so that he is much more likely to be innocent than guilty. For properly speaking there are not two arguments here but only one and the same, appealing to two circumstances, the color of clothing and the color of hair. Since these circumstances coincide only in the person of Gracchus, they argue with certainty that only he could be the murderer.

Another example. There is a question whether or not a certain written contract has been fraudulently predated. The argument for the negative might be that the document has been signed by a notary, that is, a sworn public official, who is not likely to have committed any fraud since he could not have done so without seriously jeopardizing his fortune and honor (and for this reason not one notary in fifty is found who would dare to commit such a malfeasance). The arguments for the affirmative might be that this notary has a very bad reputation, that he stood to make a lucrative profit from the fraud, and especially that he is attesting to things that have no probability (as if, for example, he had written that a certain person had entrusted 10,000 gold pieces to another at a time when, by the judgment of all, altogether his possessions could not have been worth more than 100). Here, if you were to consider by itself the argument from the character of the person who signed the document, you might judge the probability of the **[223]** document's authenticity as if it were worth $^{49}/_{50}$ of certainty. But if you weigh the arguments to the contrary, you will be forced to admit that it is hardly possible that the document has not been falsified, and to consider the commission of fraud morally certain, meaning that it has for example $^{999}/_{1000}$ of certainty. We should not, however, use the method of section 7 to conclude from this that the probability of authenticity and the probability of fraud are in a ratio of $^{49}/_{50}$ to $^{999}/_{1000}$, that is, that they are in a ratio close to equality. For when I posit that the fidelity of the notary is in disrepute, by this very fact I posit that he is not to be included with the case of the 49 honest notaries who abhor fraud, but that he is himself the 50th who does not take his oath to heart and conducts himself faithlessly in office. This is what completely obviates or destroys the force of the argument that might otherwise have proved the authenticity of the document.

Chapter IV. *On a double method of finding the numbers of cases. How the method based on experiment should be understood. A remarkable problem posed concerning this method, etc.*

It was shown in the preceding chapter how, from the numbers of cases in which arguments for things can exist or not exist, indicate or not indicate, or also indicate the contrary, and from the forces of proving proportionate to them, the probabilities of things can be reduced to calculation and evaluated. From this it resulted that the only thing needed for correctly forming conjectures on any matter is to determine the numbers of these cases accurately and then to determine how much more easily some can happen than others. But here we come to a halt, for this can hardly ever be done. Indeed, it can hardly be done anywhere except in games of chance. The originators of these games took pains to make them equitable by arranging that the numbers of cases resulting in profit or loss be definite and known and that all the cases happen equally easily. But this by no means takes place with most other effects that depend on the operation of nature or on human will. So, for example, **[224]** the numbers of cases in dice are known: for a single die there are manifestly as many cases as the die has faces. Moreover these all have equal tendencies to occur; because of the similarity of the faces and the uniform weight of the die, there is no reason why one of the faces should be more prone to fall than another—as would be the case if the faces had dissimilar shapes or if a die were composed of heavier material in one part than another. In the same way the numbers of cases for drawing white or black slips of paper from an urn are known. It is also known that they are all equally possible, because, without doubt, the number of slips of each type is known and determined and there is no reason why one of them should be drawn from the urn rather than another.

But what mortal, I ask, may determine, for example, the number of diseases, as if they were just as many cases, which may invade at any age the innumerable parts of the human body and which imply our death? And who can determine how much more easily one disease may kill than another—the plague compared to dropsy, dropsy compared to fever? Who, then, can form conjectures on the future state of life and death on this basis? Likewise who will count the innumerable cases of the changes to which the air is subject every day and on this basis conjecture its future constitution after a month, not to say after a year? Again, who has a sufficient perspective on the nature of the human mind or on the wonderful structure of the body so that they would dare to determine the cases in which this or that player may win or lose in games that depend in whole or in part on the shrewdness or the agility of the players? In these and similar situations, since they may depend on causes that are entirely hidden and that would forever mock our diligence by an innumerable variety of combinations, it would clearly be mad to want to learn anything in this way.

Nevertheless, another way is open to us by which we may obtain what is sought. What cannot be ascertained a priori, may at least be found out a posteriori from the results many times observed in similar situations, since it should be presumed that something can happen or not happen in the future in as many cases as it was observed to happen or not to happen in similar circumstances in the past. If, for example, there once existed three hundred people of the same age and body type as Titius now has, and you observed that two hundred of them died before the end of a decade, while the rest **[225]** lived longer, you could safely enough conclude that there are twice as many cases in which Titius also may die within a decade as there are cases in which he may live beyond a decade. Likewise if someone for several years past should have observed the weather and noted how many times it was clear or rainy or if someone should have very frequently watched two players at a game and should have seen how many times this or that player won, just by doing so one would have discovered the ratio that probably exists between the numbers of cases in which the same outcomes can happen or not happen in the future in circumstances similar to the previous ones.

This empirical way of determining the number of cases by experiments is neither new nor uncommon. The author of *The Art of Thinking*,[7] a man of great acuteness and talent, made a similar recommendation in Chapter 12 and following of the last part [Part IV], and everyone consistently does the same thing in daily practice. Neither should it escape anyone that to judge in this way concerning some future event it would not suffice to take one or another experiment, but a great abundance of experiments would be required, given that even the most foolish person, by some instinct of nature, alone and with no previous instruction (which is truly astonishing), has discovered that the more observations of this sort are made, the less danger there will be of error. But although this is naturally known to everyone, the demonstration by which it can be inferred from the principles of the art [of conjecturing] is hardly known at all, and, accordingly, it is incumbent upon us to expound it here. But I would consider that I had not achieved enough if I limited myself to demonstrating this one thing, of which no one is ignorant. Something else remains to think about, which perhaps no one has considered up to this point. It remains, namely, to ask whether, as the number of observations increases, so the probability increases of obtaining the true ratio between the numbers of cases in which some event can happen and not happen, such that this probability may eventually exceed any given degree of certainty. Or whether, instead, the problem has an asymptote, so to speak; whether, that is, there is some degree of certainty that may never be exceeded no matter how far the number of observations is multiplied, so that, for example, we may never be certain that we have discovered the true ratio of cases with more than a half or two-thirds or three-fourths parts of certainty.

To give an example of what I have in mind, suppose **[226]** that there are hidden in an urn, unknown to you, three thousand white tokens and two thousand black, and that, in order to investigate their number by experiments, you take out one token after another (but each time putting back the one that you have taken out before you choose the following one, lest the number of tokens in the urn be diminished) and you observe how many times a white token comes out and how many times a black one. It is asked whether you can do this so many times that it becomes ten, a hundred, a thousand, etc. times more probable (that is, that in the end it becomes morally certain) that the numbers of times in which you have chosen a white and in which you have chosen a black will have to each other the same ratio of three to two that the numbers of tokens or of cases secretly enjoy than some other different ratio. Unless, indeed, this happens, I confess it will be

7. That is, Antoine Arnauld in the well-known Port Royal logic.

all over with our effort to investigate the numbers of cases by experiments. But if it does happen and if in the end moral certainty is acquired in this way (how this will actually happen I will show in the following chapter), we will have found the numbers of cases a posteriori almost as certainly as if they were known to us a priori. This, surely, in the practice of civil life (where, by Axiom 9 of Chapter 2, what is morally certain is taken as absolutely certain), more than suffices for directing our conjectures in any contingent matter no less scientifically than in games of chance. And indeed, if in place of the urn we substitute, for example, the air or a human body, which contain within themselves the germ [*fomitem*] of various changes in the weather or diseases just as an urn contains tokens, we will be able in just the same way to determine by observation how much more easily in these subjects this or that event may happen.

Lest, however, these things be misunderstood, it must be carefully noted that we do not wish the ratio between the numbers of cases that we have undertaken to determine by experiments to be taken precisely or as an indivisible (for, if it were, then the opposite would occur, and it would become *less* probable that the true ratio had been found as more observations were taken). Rather, the ratio should be defined within some range, that is, contained within two limits, which can be made as narrow as anyone might want. Indeed, if, in the example of the tokens just discussed, we take two ratios, say $^{301}/_{200}$ and $^{299}/_{200}$, or $^{3001}/_{2000}$ and $^{2999}/_{2000}$, etc., of which one is just larger and the other just smaller than the ratio of three to two, it may be shown that it can become more probable than any given probability that the ratio found by many repeated experiments **[227]** will fall within these limits around the ratio of three to two rather than outside.

This, therefore, is the problem that I have proposed to publish in this place, after I have already concealed it for twenty years. Both its novelty and its great utility combined with its equally great difficulty can add to the weight and value of all the other chapters of this theory. But before I convey its solution, let me remove a few objections that certain learned men have raised.[8]

1. They object first that the ratio of tokens is different from the ratio of diseases or changes in the air: the former have a determinate number, the latter an indeterminate and varying one. I reply to this that both are posited to be equally uncertain and indeterminate with respect to our knowledge. On the other hand, that either is indeterminate in itself and with respect to its nature can no more be conceived by us than it can be conceived that the same thing at the same time is both created and not created by the Author of nature: for whatever God has done, God has, by that very deed, also determined at the same time.

8. These arguments appear in very similar form in correspondence from Leibniz to Bernoulli.

2. Second, they object that the number of tokens is finite, but the number of diseases etc., infinite. *Response:* It is more nearly astonishingly large than infinite, but even if it were actually infinite, it is known that there can be a determinate ratio even between two infinites, a ratio that can be expressed by finite numbers, either exactly or at least as closely any anyone might want. Certainly, the circumference of a circle has a determinate ratio to its diameter. Even if this ratio cannot be expressed accurately except by the cyclic numbers of Ludolph continued in infinitum, nevertheless it has been defined within limits sufficiently narrow for use by Archimedes, Metio, and Ludolph himself. Whence nothing prevents the ratio between two infinites from being expressed by finite numbers very closely and also determined by finite numbers of experiments.

3. Third, they add that the number of diseases does not always remain the same, but that new diseases spring up daily. *Response:* We cannot deny that the numbers of diseases multiply with the passage of time, and anyone who wanted to draw an inference from today's observations to the antediluvian times of the Patriarchs would surely stray very far from the truth. But from this it only follows that new **[228]** observations should be made in the meanwhile, just as would happen with the tokens, if their numbers in the urn were assumed to change.

Chapter V. *Solution of the preceding problem*

So that I may bring forth the body of a lengthy demonstration with the greatest possible brevity and clarity, I shall undertake to reduce it all to abstract mathematics, separating out from it the following lemmas. Once these have been shown, the rest will be simple application.

Lemma 1. Assume a series of numbers 0, 1, 2, 3, 4, etc. starting from nothing or zero and following in natural order, of which the last and greatest is called $r + s$, and some intermediate number r, and the numbers that surround it immediately on either side $r + 1$ and $r - 1$. If this series is continued onward until the farthest term whatsoever is some multiple of the number $r + s$, for instance $nr + ns$, and if the intermediate r and the adjacent numbers $r + 1$ and $r - 1$ are increased in the same ratio, so that in their places appear nr, $nr + n$, and $nr - n$, then the series itself posited at the beginning:

$$0, 1, 2, 3, 4, \ldots, r-1, r, r+1, \ldots, r+s$$

will be changed into

$$0, 1, 2, 3, 4, \ldots, nr-n, \ldots, nr, \ldots, nr+n, \ldots, nr+ns.$$

In this way, indeed, the terms of the series will be multiplied, both those that come between the middle nr and either of its bounds $nr + n$ or $nr - n$, and those that extend onward from these bounds to the outermost terms, $nr + ns$ or 0. Never, however (no matter how large n is assumed to be), will the number of terms beyond the larger bound $nr + n$ exceed more than $s - 1$ times, nor will the number of terms beyond the smaller bound $nr - n$ exceed more than $r - 1$ times, the number of those terms that are included between the middle term nr and either bound, $nr + n$ or $nr - n$. For when the subtraction is made, it is clear that from the larger bound to the extreme bound $nr + ns$ there is an interval of $ns - n$ terms; and from the smaller bound to the other extreme 0 there is an interval of $nr - n$ terms; and from the intermediate number to either bound an interval of n terms. And, indeed, it is always true that $(ns - n) : n :: (s - 1) : 1$; and $(nr - n) : n :: (r - 1) : 1$, whence it is established, etc. **[229]**

Lemma 2. Every integral power of a binomial $r + s$ is expressed by one more term than the number of units in the index of the power. Thus a square is composed of 3 terms, a cube of 4, a biquadrate of 5, and so forth, as is known.

Lemma 3. In any power of this binomial (at least in any power of which the index is equal to the binomial $r + s = t$, or to a multiple of it, that is, $nr + ns = nt$), if some terms precede and others follow some term M, such that the number of all the preceding terms to the number of all the following terms is, reciprocally, as s to r (or, equivalently, if in that term the numbers of dimensions of the letters r and s are directly as the quantities r and s themselves), then that term will be the largest of all the terms in that power, and the terms nearer it on either side will be larger than the terms farther away on the same side. But this same term M will have a smaller ratio to the terms closer to it than those nearer terms (in an equal interval of terms) have to the farther terms.

Demonstration. 1. It is known among mathematicians that the nt power of the binomial $r + s$, that is, $(r + s)^{nt}$, is expressed by this series:

$$r^{nt} + \frac{nt}{1} r^{nt-1}s + \frac{nt(nt-1)}{1 \cdot 2} r^{nt-2}ss + \frac{nt(nt-1)(nt-2)}{1 \cdot 2 \cdot 3} r^{nt-3}s^3 + \ldots$$
$$+ \frac{nt}{1} rs^{nt-1} + s^{nt};$$

In this progression one part of the binomial, namely r, decreases its dimensions gradually, while the other part, s, increases its dimensions. Moreover, the coefficients of the second and second to last terms are $nt/1$, of the third and third to last terms $nt(nt-1)/(1 \cdot 2)$, of the fourth and fourth to last $nt(nt-1)(nt-2)/(1 \cdot 2 \cdot 3)$, and so forth. And because the number of all the terms except M, by Lemma 2, is $nt = nr + ns$, but, by hypothesis, the number of terms preceding M to the number of terms following it is as s to r, it follows that the number of terms preceding term M will be ns, and the number following it will be nr. Whence, from the law of the progression, the term M becomes

$$\frac{nt(nt-1)(nt-2)\ldots(nt-ns+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots ns} r^{nr}s^{ns} = \frac{nt(nt-1)(nt-2)\ldots(nr+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots ns} r^{nr}s^{ns},$$

or

$$\frac{nt(nt-1)(nt-2)\ldots(nt-nr+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots nr} r^{nr}s^{ns} = \frac{nt(nt-1)(nt-2)\ldots(ns+1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots nr} r^{nr}s^{ns}.$$

Similarly the adjacent terms are, **[230]**

to the left: $$\frac{nt(nt-1)(nt-2)\ldots(nr+2)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots (ns-1)} r^{nr+1}s^{ns-1}$$

to the right: $$\frac{nt(nt-1)(nt-2)\ldots(ns+2)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots (nr-1)} r^{nr-1}s^{ns+1},$$

and the following are, to the left: $$\frac{nt(nt-1)(nt-2)\ldots(nr+3)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots (ns-2)} r^{nr+2}s^{ns-2}$$

to the right: $$\frac{nt(nt-1)(nt-2)\ldots(ns+3)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots (nr-2)} r^{nr-2}s^{ns+2},$$

from which, once an appropriate reduction has been made both of the coefficients and of the pure terms, by common divisors, it follows that the ratio of the term M to the next term to the left is as $(nr+1)s$ to $ns \cdot r$, and the latter to the following, as $(nr+2)s$ to $(ns-1)r$, etc. Similarly, the term M to the next term to the right will be as $(ns+1)r$ to $nr \cdot s$; and the latter to the following, as $(ns+2)r$ to $(nr-1)s$, etc. But $(nr+1)s$ [$= nrs + s$] $> ns \cdot r$ [$= nrs$], and $(nr+2)s$ [$= nrs + 2s$] $> (ns-1)r$ [$= nrs - r$], etc. Moreover, $(ns+1)r$ [$= nrs + r$] $> nr \cdot s$ [$= nrs$], and $(ns+2)r$ [$= (nrs + 2r)$] $> (nr-1)s$ [$= nrs - s$], etc., as is clear. Therefore the term M is larger than the next term to either side, the latter is larger than the farther one on the same side, etc. Q.E.D.

2. The ratio $(nr+1)/ns$ is smaller than the ratio $(nr+2)/(ns-1)$, as is clear. Therefore when the common ratio s/r is *added* [*addita*], $(nr+1)s/(ns \cdot r) < (nr+2)s/(ns-1)r$. Similarly, the ratio $(ns+1)/nr < (ns+2)/(nr-1)$, as is transparent. Thus when the common ratio r/s is *added*, so too the ratio $(ns+1)r/$

$(nr \cdot s) < (ns + 2)r/(nr - 1)s\}$. But $(nr + 1)s/(ns \cdot r)$ is the ratio that the term M has to the next term to the left; and $(nr + 2)s/(ns - 1)r$ is the ratio which the latter has to the following. Likewise, $(ns + 1)r/(nr \cdot s)$ is the ratio that the term M has to the next term to the right; and $(ns + 2)r/(nr - 1)s$ is the ratio that the latter has to the following term as has just been shown. The same can be concluded *ex aequo* for all the others. Whence the maximum term M has to a term nearer it on either side a smaller ratio than (in a equal interval of terms) the nearer has to a further on the same side. Q.E.D. **[231]**

Lemma 4. In a power of a binomial with index nt, the number n can be conceived to be so large that the largest term M acquires a ratio to the terms L and Λ, which are at an interval of n terms to the left and right of it that is larger than any given ratio.

Demonstration. Since in the preceding lemma the term M was found to be

$$\frac{nt(nt-1)(nt-2)\ldots(nr+1)}{1\cdot 2\cdot 3\cdot 4\ldots ns}\, r^{nr}s^{ns},$$

or

$$\frac{nt(nt-1)(nt-2)\ldots(ns+1)}{1\cdot 2\cdot 3\cdot 4\ldots nr}\, r^{nr}s^{ns},$$

it follows from the law of the progression (adding n to the last factor of the coefficient in the numerator and subtracting it from the last factor in the denominator; also increasing one of the letters r and s by n dimensions and decreasing the other) that the term

L on the left is: $$\frac{nt(nt-1)(nt-2)\ldots(nr+n+1)}{1\cdot 2\cdot 3\cdot 4\ldots(ns-n)}\, r^{nr+n}s^{ns-n}$$

Λ on the right is: $$\frac{nt(nt-1)(nt-2)\ldots(ns+n+1)}{1\cdot 2\cdot 3\cdot 4\ldots(nr-n)}\, r^{nr-n}s^{ns+n};$$

whence, when the appropriate reduction by common divisors is made, there results:

$$\frac{M}{L} = \frac{(nr+n)(nr+n-1)(nr+n-2)\ldots(nr+1)\times s^n}{(ns-n+1)(ns-n+2)(ns-n+3)\ldots ns\times r^n}$$

$$\frac{M}{\Lambda} = \frac{(ns+n)(ns+n-1)(ns+n-2)\ldots(ns+1)\times r^n}{(nr-n+1)(nr-n+2)(nr-n+3)\ldots nr\times s^n}$$

or, if the dimensions of the quantities r^n and s^n are evenly distributed among the individual factors (there being the same number of dimensions as factors),

$$\frac{M}{L}=\frac{(nrs+ns)(nrs+ns-s)(nrs+ns-2s)\ldots nrs+s}{(nrs-nr+r)(nrs-nr+2r)(nrs-nr+3r)\ldots nrs}$$

$$\frac{M}{\Lambda}=\frac{(nrs+nr)(nrs+nr-r)(nrs+nr-2r)\ldots(nrs+r)}{(nrs-ns+s)(nrs-ns+2s)(nrs-ns+3s)\ldots nrs}.$$

But these ratios are infinitely large when the number n is posited to be infinite. Then the numbers 1, 2, 3, etc. vanish before n and $nr \pm n \mp 1, 2, 3,$ etc. and $ns \mp n \pm 1, 2, 3,$ etc. have the same value as $nr \pm n$ and $ns \mp n$. After division by n, we have: **[232]**

$$\frac{M}{L}=\frac{(rs+s)(rs+s)(rs+s)\ldots rs}{(rs-r)(rs-r)(rs-r)\ldots rs}$$
$$\frac{M}{\Lambda}=\frac{(rs+r)(rs+r)(rs+r)\ldots rs)}{(rs-s)(rs-s)(rs-s)\ldots rs)}.$$

These quantities are compounded, clearly, from as many ratios $(rs+s)/(rs-r)$ or $(rs+r)/(rs-s)$ as there are factors. But the number of factors is n, that is, infinite, since between the first, $nr+n$ or $ns+n$, and the last, $nr+1$ or $ns+1$, the difference is $n-1$. Therefore, these ratios are the ratios $(rs+s)/(rs-r)$ and $(rs+r)/(rs-s)$ multiplied times themselves infinitely many times [*infinituplicatae*][9] or, accordingly, simply infinite. If you doubt this consequence, consider infinitely many things continuously proportional in the ratio of $rs+s$ to $rs-r$, or $rs+r$ to $rs-s$. Then the first to the third will have the ratio *duplicata,* the first to the fourth in the ratio *triplicata,* to the fifth in the ratio *quadruplicata,* etc. and to the last in the ratio $(rs+s)/(rs-r)$ or $(rs+r)/(rs-s)$ infinitely many times itself [*infinituplicata*]. It is established, however, that the ratio of the first to the last is infinitely large, because the last = 0 (see the Corollary to our Sixth Proposition of our *De Seriebus Infinitis*).[10] Therefore it is also established that the ratio $(rs+s)/(rs-r)$ or $(rs+r)/(rs-s)$ infinitely many times itself is infinite. Consequently, it has

9. In this and the following sentences Bernoulli uses the classical and medieval terms, which distinguish operations on ratios from those on fractions. In the following sentences *duplicata* is equivalent to "squared" in modern terms, *triplicata* is "cubed," and *quadruplicata* is "to the fourth power."

10. This appears on p. 244 in the same volume: "*VI. In Progress. Geometr. decrescente A, B, C, D, E pervenitur tandem ad terminum E quovis dato Z minorem. . . . Coroll.* Hinc in Progr. Geomet. decrescente in infinitum continuata ultimus terminus est 0, per Prop. 1."

been shown that, in an infinitely high power of the binomial, the maximum term M to the terms L and Λ has a ratio larger than any assignable ratio. Q.E.D.

Lemma. 5. Given what has been posited in the preceding lemmas, n can be taken to be so large that the sum of all the terms between the middle and maximum term M and the bounds L and Λ inclusively has to the sum of all the remaining terms outside the bounds L and Λ a ratio larger than any given ratio.

Demonstration. Of the terms between the maximum M and the bound to the left L, let the second from the maximum be called F, the third G, the fourth H, and so forth; and outside the bound let the second from it be called P, the third Q, the fourth R, and so forth. Then since $M/F < L/P$, $F/G < P/Q$, and $G/H < Q/R$, etc., by the second part of Lemma 3, it follows also that $M/L < F/P < G/Q < H/R$, etc. Since, when n is taken to be an infinite number, the ratio M/L is infinitely large, by [233] Lemma 4, a fortiori the other ratios F/P, G/Q, H/R, etc. will also be infinite; and, because of this, $(F + G + H + \text{etc.})/(P + Q + R + \text{etc.})$ will also be infinite, that is, all the terms contained between the maximum M and the bound L will be infinitely greater than just as many terms extending immediately beyond L. And since the number of all the terms outside the bound L does not exceed all the terms between the bound and the maximum M more than $s - 1$ times (that is, only a finite number of times), by Lemma 1, and since these terms become smaller the farther they are from the bound, by the first part of Lemma 3, therefore all the terms together between M and L (even if M is not counted) will infinitely exceed all the terms beyond L taken together. It may be shown similarly, on the other side, that all the terms included between M and Λ are infinitely greater than all the terms extending beyond Λ (of which the number, by Lemma 1, does not exceed the number between M and Λ by more than $r - 1$ times). Whence, finally, all the terms contained between the two bounds L and Λ (even if you exclude the maximum M) will similarly infinitely exceed all the terms altogether outside, and all the more so when the maximum term is included. Q.E.D.

Scholium. It may be objected against Lemmas 4 and 5, by those who are not accustomed to speculations about the infinite, that in the case of an infinite number n, even if the factors in the expressions for the ratios M/L and M/Λ, namely $nr \pm n \mp 1, 2, 3$, etc., and $ns \mp n \pm 1, 2, 3$ etc., have the same value as $nr \pm n$ and $ns \mp n$, because the numbers 1, 2, 3, etc. vanish from the individual factors in the ratio, it could nevertheless happen that, when all of these are taken together or multiplied into each other, they might (on account of the infinite number of factors), grow *in infinitum,* thus infinitely diminishing—that is, rendering finite—

the ratio equal to the ratio $(rs + s)/(rs - r)$ or $(rs + r)/(rs - s)$ multiplied into itself infinitely many times. I cannot reply to this uneasiness better than by showing how to assign an actually finite number to n, or a finite power to the binomial, so that the sum of the terms within the bounds L and Λ will have to the sum of terms outside a ratio larger than any given ratio however large—which I designate by the letter c. When this has been shown, it will be seen that the objection necessarily also collapses. **[234]**

To this end I take any ratio of greater inequality, which, however, is less than the ratio $(rs + s)/(rs - r)$ (for the terms to the left side), for example, the ratio $(rs + s)/rs$ or $(r + 1)/r$, and I multiply it times itself sufficiently many times (say m times) to make it equal to or greater than the ratio of $c(s - 1)$ to 1, that is, so that $(r + 1)^m/r^m \geq c(s - 1)$. How many times that should be can quickly be ascertained using logarithms. Taking the logarithms of the quantities yields $m \log (r + 1) - m \log r \geq \log [c(s - 1)]$. Dividing, I immediately have $m \geq \log [c(s - 1)]/[\log (r + 1) - \log r]$.

Once this is found, I continue: in the series of fractions or factors $(nrs + ns)/(nrs - nr + r) \cdot (nrs + ns - s)/(nrs - nr + 2r) \cdot (nrs + ns - 2s)/(nrs - nr + 3r) \ldots (nrs + s)/nrs$, the multiplication of which, by Lemma 4, yields the ratio M/L, it may be observed that the individual fractions are less than $(rs + s)/(rs - r)$, in such a way that they continually approach nearer to it as n is taken larger. Consequently, any of them at some time becomes equal to $(rs + s)/rs = (r + 1)/r$. So let us see how great a value of n must be taken so that the fraction whose position in the series is m is equal to $(r + 1)/r$. It is clear from the law of the progression that the fraction in the mth position is: $(nrs + ns - ms + s)/(nrs - nr + mr)$, which, equated to the fraction $(r + 1)/r$, gives $n = m + (ms - s)/(r + 1)$, whence $nt = mt + (mst - st)/(r + 1)$. I say that this is the index of the power to which the binomial $r + s$ should be raised for the maximum term M to exceed the bound L more than $c(s - 1)$ times. Now because the fraction in the mth position, given this value of the number n, becomes equal to $(r + 1)/r$, and by hypothesis the fraction $(r + 1)/r$ multiplied by itself m times, that is, $(r + 1)^m/r^m$, by construction equals or exceeds $c(s - 1)$, **[235]** it follows that this fraction multiplied times all the preceding fractions will exceed $c(s - 1)$ even more, since each preceding fraction is greater than $(r + 1)/r$. Therefore the result will exceed $c(s - 1)$ even more when it is multiplied together as before with all the following fractions, since each of them at least exceeds a ratio of equality. But the product of all these fractions is the ratio of M to L; therefore it is completely established that the term M exceeds the bound L more than $c(s - 1)$ times. But it has already been shown that $M/L < F/P < G/Q < H/R$. Hence, by still more the second term from the maximum M will exceed the second from the bound L by more than $c(s - 1)$ times, and by still more the third will exceed the third, etc. So in the end all the terms between the maximum M and the bound L will exceed more than $c(s-1)$ times the same number of terms starting from the largest terms outside this bound. Similarly, they will exceed more than c times that many terms taken $s-1$ times. Therefore, even more obviously, they will exceed more than c times all the terms outside the bound L, of which there are only $s-1$ times as many more.

For the terms to the right, I proceed in the same way. I take the ratio $(s+1)/s < (rs+r)/(rs-s)$, I set $(s+1)^m/s^m \geq c(r-1)$, and I find that $m \geq \log [c(r-1)]/[\log (s+1) - \log s]$. Next, in the series of fractions $(nrs+nr)/(nrs-ns+s) \cdot (nrs+nr-r)/(nrs-ns+2s) \cdot (nrs+nr-2r)/(nrs-ns+3s) \ldots (nrs+r)/nrs$, which signify the ratio M/Λ, I suppose that the fraction in the mth position, namely $(nrs+nr-mr+r)/(nrs-ns+ms)$, is equal to $(s+1)/s$, and from this I find that $n = m + (mr-r)/(s+1)$, and hence that $nt = mt + (mrt-rt)/(s+1)$. This having been done, it is similarly shown, as before, that, when the binomial $r+s$ is taken to this power, its maximum term M exceeds the bound Λ more than $c(r-1)$ times. Consequently also, all the terms included between the maximum M and the bound Λ exceed by more than c times all the terms outside this bound, of which there are only $r-1$ times more. Thus finally, in the end, we conclude that when the binomial $r+s$ is raised to the power of which the index is equal to the larger of these two quantities, **[236]** $mt + (mst-st)/(r+1)$ and $mt + (mrt-rt)/(s+1)$, then the sum of the terms included between the two bounds L and Λ exceeds by much more than c times the sum of the terms beyond the bounds on both sides. Therefore a finite power has been found that has the desired property. Q.E.D.

Principal Proposition. Finally, there follows the proposition for the sake of which all this has been said, but whose demonstration can now be given with only the application of the foregoing lemmas. To avoid tedious circumlocution, I will call the cases in which a certain event can happen *fecund* or *fertile.* I will call *sterile* those cases in which the event can not happen. I will also call experiments *fecund* or *fertile* in which one of the fertile cases is discovered to occur; and I will call *nonfecund* or *sterile* those in which one of the sterile cases is observed to happen. Let the number of fertile cases and the number of sterile cases have exactly or approximately the ratio r/s, and let the number of fertile cases to all the cases be in the ratio $r/(r+s)$ or r/t, which ratio is bounded by the limits $(r+1)/t$ and $(r-1)/t$. It is to be shown that so many experiments can be taken that it becomes any given number of times (say c times) more likely [*verisimilius*] that the number of fertile observations will fall between these bounds than outside them, that is, that the ratio of the number of fertile to the number of all the observations will have a ratio that is neither more than $(r+1)/t$ nor less than $(r-1)/t$.

Demonstration. Let nt be the number of observations to be taken, and let us ask how great is the expectation or how great is the probability that they will all be fecund except for, first, none, then 1, 2, 3, 4, etc. sterile. But since in any observation I exhibit there are, by hypothesis, t cases, and of them r are fecund and s sterile, and the single cases of one observation can be combined with the single cases of the other, and those combined can be joined again with the single cases of the third, fourth, etc., it is easy to see that this situation fits the Rule in the Notes appended to the end of Proposition **[237]** XIII [*sic,* should be XII] in Part I, and its Corollary 2, which contains a general formula, with the help of which it is seen that the expectation of no sterile observations is $r^{nt} : t^{nt}$, of one $(nt/1)r^{nt-1}s : t^{nt}$, of two $[nt(nt-1)/(1 \cdot 2)]r^{nt-2}ss : t^{nt}$, of three $[nt(nt-1)(nt-2)/(1 \cdot 2 \cdot 3)]r^{nt-3}s^3 : t^{nt}$, and so forth. Consequently, omitting the common denominator t^{nt}, the degrees of probability or the numbers of cases in which it can happen that all the experiments are fecund, or all except 1 sterile one, or all except 2, 3, 4, etc. are expressed in order by

$$r^{nt}, \frac{nt}{1} r^{nt-1}s, \frac{nt(nt-1)}{1 \cdot 2} r^{nt-2}ss, \frac{nt(nt-1)(nt-2)}{1 \cdot 2 \cdot 3} r^{nt-3}s^3, \text{ etc.}$$

Now these, in fact, are the terms of the power nt of the binomial $r + s$, investigated just now in our lemmas. Then all the rest is completely evident.

Indeed, it is clear from the nature of the progression that the number of cases that combine ns sterile experiments with nr fecund ones is the maximum term M, or the term that ns terms precede and nr follow, by Lemma 3. It is also clear that the numbers of those cases in which either $nr + n$ or $nr - n$ fecund experiments occur with the others sterile are represented by the terms L and Λ of the power, which are n terms from the maximum term M on either side. Consequently, the sum of the cases in which not more than $nr + n$ nor fewer than $nr - n$ experiments happen to be fecund is expressed by the sum of the terms of the power contained between the bounds L and Λ. The sum of the remaining cases, in which either more or fewer fecund experiments are found, is expressed by the sum of the other terms that are beyond the bounds L and Λ. Whence, since the power of the binomial can be taken to be so great that the sum of the terms included between the bounds L and Λ exceeds more than c times the sum of the others beyond these bounds, by Lemmas 4 and 5 it follows that enough observations can be taken that the sum of the cases in which the number of fertile observations happens to have a ratio to the number of all observations that is within the bounds $(nr + n)/nt$ and **[238]** $(nr - n)/nt$, or $(r + 1)/t$ and $(r - 1)/t$, will exceed the sum of the remaining cases by more than c times. In other words, it is rendered more than c times more probable that the ratio of the number of fertile observations to the number of all the observations will fall within the bounds $(r + 1)/t$ and $(r - 1)/t$ than that it will fall outside. Q.E.D.

In the application of numbers to these results, it is self-evident that the larger the numbers r, s, and t in the same ratio to each other, the more the bounds $(r + 1)/t$ and $(r - 1)/t$ to the ratio r/t can be tightened. Therefore, if the ratio between the numbers of cases r/s, to be determined by experiments, is, say, a three-halves ratio, I do not use 3 and 2 for r and s, but rather 30 and 20, or 300 and 200, etc. It might be sufficient to set $r = 30$, $s = 20$, and $t = r + s = 50$, so that the bounds become $(r + 1)/t = 31/50$, and $(r - 1)/t = 29/50$. Moreover, let $c = 1000$. Then by the preceding in the Scholium, for the terms to

the left: $$m > \frac{\log [c(s-1)]}{\log (r+1) - \log r} = \frac{4.2787536}{142405} < 301$$

$$nt = mt + \frac{mst - s}{r+1} < 24{,}728$$

the right: $$m > \frac{\log [c \cdot (r-1)]}{\log (s+1) - \log s} = \frac{4.4623980}{211893} < 211$$

$$nt = mt + \frac{mrt - rt}{s+1} = 25{,}550.$$

Whence, by what has been demonstrated, it is inferred that if 25,550 experiments are taken, it will be more than 1000 times more likely [*verisimilius*] that the ratio of the number of fertile observations to the number of all the observations will fall between these bounds, 31/50 and 29/50, than outside them. On the same understanding, if c is set equal to 10,000 or 100,000, it may be seen that it will be more than ten thousand times more probable, if there are 31,258 experiments, and more than a hundred thousand times more probable, **[239]** if there are 36,966, and so forth to infinity, continually adding to the 25,550 another 5708 experiments. Whence at last this remarkable result is seen to follow, that if the observations of all events were continued for the whole of eternity (with the probability finally transformed into perfect certainty) then everything in the world would be observed to happen in fixed ratios and with a constant law of alternation. Thus in even the most accidental and fortuitous we would be bound to acknowledge a certain quasi-necessity and, so to speak, fatality. I do not know whether or not Plato already wished to assert this result in his dogma of the universal return of things to their former positions [*apocatastasis*], in which he predicted that after the unrolling of innumerable centuries everything would return to its original state.