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Rough guide to spontaneous symmetry breaking

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Understanding spontaneous symmetry breaking is essential to understanding characteristic phenomena of solid-state physics, condensed matter physics, elementary particle physics, and cosmology. I will forego the details of physical applications in order to concentrate on the implications of spontaneous symmetry breaking for general issues concerning laws and symmetries and for the foundations of quantum field theory (QFT).[1]

1 Laws and symmetries

The most intriguing aspects of spontaneous symmetry breaking derive from features peculiar to QFT. But before turning to this theory, I want to rehearse some themes that are familiar from that innocent era that existed not only prior to the advent of QFT but also before the *verdammt* quantum complicated our world.

Theme 1. Consider equations of motion that are derived from an action principle that demands that the allowed motions extremize the action $\mathfrak{A} = \int_\Omega L(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(n)})\, d^p\mathbf{x}$ (here $\mathbf{x} = (x^1, \ldots, x^p)$ stands for the independent variables, $\mathbf{u} = (u^1, \ldots, u^r)$ are the dependent variables, and the $\mathbf{u}^{(n)}$ are derivatives of the dependent variables up to some finite order n with respect to the x^i); the resulting equations of motion take the form of Euler–Lagrange equations. This is a mild restriction in the sense that it covers almost all of the live candidates for the role of fundamental equations of motion in modern physics. A Lie group $\mathcal{G}$ of transformations $\mathcal{G} \ni g : (\mathbf{x}, \mathbf{u}) \rightarrow (\mathbf{x}', \mathbf{u}')$ is called a *variational symmetry group* if the infinitesimal generators of $\mathcal{G}$ leave L form invariant. Such symmetries are necessarily symmetries of the Euler–Lagrange equations in that they carry solutions to solutions.[2]

[1] For a more detailed guide, see J. Earman, 'Spontaneous symmetry breaking for philosophers', unpublished manuscript. Good reviews of the physics of spontaneous symmetry breaking abound in the physics literature; I can especially recommend Aitchison (1982, chapters 5 and 6), Coleman (1985, chapter 5), and Guralnik *et al.* (1968).

[2] The converse is not true.

Furthermore, if $\mathcal{G}$ is an s-parameter ($s < \infty$) Lie group, then Noether's first theorem tells us that there are s conserved currents.[3] In the cases to be considered below, the transformations $\mathcal{G}$ are such as to produce a unit Jacobian for the transformation of the independent variables and, thus, produce strict numerical invariance of the action. Such symmetries are referred to somewhat imprecisely in the physics literature as 'symmetries of the Lagrangian'. I will follow this usage here.

Theme 2. Symmetries of the laws of motion need not be – and typically are not – exhibited in particular solutions or in particular states belonging to solutions.

Combining these two themes yields the moral that a symmetry of the Lagrangian may be 'broken' in that it may not be exhibited in the solution corresponding to the actual world. For example, the tables, chairs, and other objects in my neighbourhood do not exhibit rotational symmetry even though the Lagrangian that governs the motion of the particles that compose these objects is (presumably) rotationally invariant. Put in this light, symmetry breaking seems so commonplace and so innocuous that, at first blush, it is hard to see how it could hold any interest for philosophers of physics. However, cases of symmetry breaking do serve to underscore the point – emphatically emphasized by Eugene Wigner (1967) – that an appreciation of the role of symmetry principles in physics presupposes a distinction between laws and initial/boundary conditions since the symmetries of concern to the physicist are symmetries of the former that typically fail to be symmetries of the latter. Taken at face value, this point appears to be flatly inconsistent with the so-called 'no-laws' view of science, at least insofar as this view denigrates or downplays the concept of laws of physics. It remains to be seen whether it is incompatible with the more moderate no-laws view which does not denigrate the concept of laws of physics – in the sense of general principles fashioned by us in our attempts to understand the world – but denies that these principles capture the laws of nature after which philosophers have lusted – in the sense of objective facts about nomic necessities.[4] A second methodological issue arises from the first. If asymmetry is such a pervasive feature of the phenomena, why should physicists believe so strongly in the symmetry of the laws that govern these phenomena?[5] And if their reasons are good reasons, do they provide a basis for scientific realism with respect to the unobservables in terms of which the laws governing the phenomena are formulated?

I leave it to the philosophers to deal with these methodological issues, in order to concentrate on the subclass of cases that fall under the heading of spontaneous symmetry breaking.

[3] For details see Brading and Brown, this volume.
[4] Two rather different no-laws views are to be found in van Fraassen (1989) and Giere (1999).
[5] On this matter see Kosso (2000).

2 Spontaneous symmetry breaking in classical physics and QFT

What distinguishes cases of spontaneously broken symmetries from generic cases of broken symmetries? Weinberg (1996, p. 163) casts the net widely by counting as a case of spontaneous symmetry breaking any one where the ground state (lowest energy state) is degenerate and where the symmetry of the Lagrangian is not a symmetry of the ground states but carries one asymmetric ground state to another. A more common usage would restrict the label 'spontaneous symmetry breaking' to cases where, as the value of some parameter surpasses a critical value, the ground state becomes degenerate and, without any apparent asymmetrical cause, the system enters one of the asymmetrical degenerate ground states. Examples of this kind can be given in elementary classical mechanics – a rather nice one was constructed by Greenberger (1978).[6] On closer inspection, however, it turns out that the breaking of the symmetry is not really spontaneous and that the appearance of a spontaneous breaking arises from the combination of dynamical instability plus some asymmetrical factor which may not be macroscopically detectable.

I will not pause to enter the debate about how exactly the subclass of spontaneously broken symmetries is to be delimited from the general class of broken symmetries. For my main concern is with spontaneous symmetry breaking in QFT, and in this context the situation is quite different from that in classical mechanics or ordinary non-relativistic quantum mechanics (QM). For starters, as long as the system of interest is closed, there is no temporal evolution involved in spontaneous symmetry breaking in QFT since *every* physically relevant state of the system is asymmetric with respect to the symmetry of the Lagrangian. A better term than 'spontaneously broken symmetry' might be 'ubiquitously broken symmetry'. Furthermore, although some of the words describing spontaneous symmetry breaking in classical mechanics and ordinary QM carry over to QFT, their meaning is quite different. For instance, the role of the 'ground state' is played in QFT by the vacuum state in a Fock space, and in cases of spontaneous symmetry breaking the vacuum state becomes 'degenerate'. However, the relevant sense of degeneracy in QFT is wholly different from that in classical mechanics or ordinary QM. Indeed, the situation in QFT is so unfamiliar it can generate a sense of the cognitive dissonance. Here are a few of the puzzles that someone encountering the topic for the first time may experience:

- In cases of spontaneous symmetry breaking in QFT a symmetry of the Lagrangian is spontaneously broken by failing to be a symmetry of the vacuum state of the system. But if the vacuum state is a state of nothingness, how can such a state fail to share the symmetries of the Lagrangian?

[6] See also C. Liu, 'Spontaneous symmetry breaking (I): Its meaning from a simple classical model,' preprint.

- Since the said symmetry of the Lagrangian is not a symmetry of the vacuum state of the system, it carries this state to another state which, it can turn out, is also a vacuum state of the system. So in cases of spontaneous symmetry breaking in QFT, the vacuum state is degenerate. But how can this be, since in relativistic QFT the vacuum state is supposed to be the unique state picked out by Poincaré invariance and positivity of energy?
- Part of the answer to the second puzzle is that the said symmetry of the Lagrangian is not unitarily implementable, i.e. its action is not faithfully represented by a unitary operator on Hilbert space. But how can this be, since Wigner's theorem has taught us that a symmetry in QM is represented by a unitary transformation (or, as in the case of time reversal, an anti-unitary transformation)?
- The remainder of the answer to the second puzzle involves the fact that the 'degenerate' vacuum states belong to unitarily inequivalent representations of the algebra of observables. But what does this mean physically and why does it happen in QFT and not in ordinary QM?

To help resolve these puzzles I turn to the algebraic formulation of QFT.

3 Using the algebraic formulation of QFT to explain spontaneous symmetry breaking

The algebraic formulation of QFT has the virtue of making transparent the abstract mathematical structure of cases of spontaneous symmetry breaking in QFT. It points one to definitions, both natural and precise, of the relevant senses of symmetry and symmetry breaking, and it allows the formulation and proof of quite general statements about the conditions under which a symmetry is or is not unitarily implementable. Anyone who masters this formalism will not be subject to the puzzles enumerated above.

In the algebraic approach the basic object is an abstract algebra $\mathcal{A}$, usually taken to be a C^*-algebra with unit element.[7] An algebraic state is a normed positive linear functional $\omega : \mathcal{A} \to \mathbb{C}$, and the expectation value of an element $A \in \mathcal{A}$ is then $\omega(A)$. The familiar Hilbert space formalism is recovered in a representation of $\mathcal{A}$ in the form of a structure preserving map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ into the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, which might or might not be a Fock space. A fundamental result of Gelfand, Nimark, and Segal (GNS) guarantees that any state ω determines a cyclic representation $(\pi_\omega, \mathcal{H}_\omega)$ that is unique up to unitary equivalence.

In the algebraic setting a symmetry is given by an automorphism θ of the algebra $\mathcal{A}$.[8] An automorphism θ of $\mathcal{A}$ is said to be *inner* just in case there is a unitary $U \in \mathcal{A}$

[7] See appendix. A detailed presentation of the mathematics of algebraic QFT is to be found in Bratteli and Robinson (1987; 1996).

[8] Everything said here is easily generalized to groups of automorphisms.

such that $\theta(A) = UAU^{-1}$ for every $A \in \mathcal{A}$. Many automorphisms of C^* -algebras are not inner. And worse, they may even fail to be inner with respect to a state ω in that they fail to have the property that θ behaves like an inner automorphism 'under the expectation value', i.e. there is no unitary $U \in \mathcal{A}$ such that $\omega(\theta(A)) = \omega(UAU)$ for every $A \in \mathcal{A}$. Glimm and Kadison (1960) proved that θ is inner with respect to a pure state ω iff θ is unitarily implementable with respect to ω in that in the GNS representation $(\pi_\omega, \mathcal{H}_\omega)$ determined by ω, there is a unitary operator $\hat{U}$ on $\mathcal{H}_\omega$ such that $\pi_\omega(\theta(A)) = \hat{U}\pi_\omega(A)\hat{U}^{-1}$ for every $A \in \mathcal{A}$. Typical cases of spontaneous symmetry breaking in QFT turn out to be cases where there is a symmetry of the Lagrangian that induces an automorphism θ of the algebra $\mathcal{A}$ of field operators, but θ is not unitarily implementable with respect to any physically relevant state ω (see below).

Note that an automorphism θ of the algebra $\mathcal{A}$ can be thought of as acting on states: for any state ω, θ produces a new state $\widehat{\theta\omega}$ where $\widehat{\theta\omega}(A) := \omega(\theta(A))$ for every $A \in \mathcal{A}$.[9] If ω is θ-invariant, i.e. $\widehat{\theta\omega} = \omega$, then trivially θ is unitarily implementable with respect to ω. Since in spontaneous symmetry breaking we are concerned with cases where a symmetry θ is not unitarily implementable, $\widehat{\theta\omega} \neq \omega$. If ω is a vacuum state and θ commutes with the symmetry (e.g. Poincaré transformations), the invariance under which picks out a vacuum state, then $\widehat{\theta\omega}$ will be a different vacuum state. And, moreover, these different vacuum states belong to unitarily inequivalent representations of $\mathcal{A}$, for the Glimm and Kadison result can be extended to show that θ is unitarily implementable with respect to a pure state ω iff $\widehat{\theta\omega}$ and ω determine unitarily equivalent GNS representations of $\mathcal{A}$, in that there is an isomorphism $\hat{V} : \mathcal{H}_\omega \to \mathcal{H}_{\widehat{\theta\omega}}$ such that $\pi_{\widehat{\theta\omega}}(A) = \hat{V}\pi_\omega(A)\hat{V}^{-1}$ for every $A \in \mathcal{A}$ (see Arageorgis *et al.*, 2002).

A concrete toy example of spontaneous symmetry breaking in QFT starts from the Lagrangian density $L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$ for a real-valued scalar field φ. L is invariant under the transformations of the field $\varphi \to \varphi' = \varphi + \chi$, where χ is an arbitrary real number. When the field is quantized this symmetry is not unitarily implementable. The treatment of this example from the perspective of standard QFT is to be found in Aitchison (1982, chapter 5), and the algebraic analysis is given in Streater (1965). In this particular example, as in many examples of spontaneous symmetry breaking, there is a nice connection with Noether's first theorem. In the present case, Noether's first theorem tells us that since the variational symmetries form a one-dimensional Lie group, there is a conserved current j^μ. The spatial volume integral of the time component j^0 is the Noether charge Q. If, upon quantization of the field, there were a well-defined self-adjoint charge operator $\hat{Q}$ corresponding to Q, it would be the generator of a 1-parameter family of unitary transformations that

9 If the automorphism represents time evolution, the difference between the two points of view amounts to the difference between the Heisenberg and Schrödinger pictures.

implement the transformation of the quantum field $\hat{\varphi} \to \hat{\varphi}' = \hat{\varphi} + \chi$. But a reductio argument shows that there is no such $\hat{Q}$ if the vacuum state is (as normally assumed) translationally invariant and if $\hat{Q}$ commutes with translations (see Aitchison, 1982, pp. 71–2, and Fabri and Picasso, 1966).

To translate this no-go result back into the algebraic setting, call an algebraic state ω on a C^*-algebra $\mathcal{A}$ a *Fock state* if its GNS representation is unitarily equivalent to a Fock representation. Then what the reductio proof for the toy model shows is that if it is demanded that algebraic states be Fock states with translationally invariant vacuums, there is no state ω with respect to which the automorphism θ of $\mathcal{A}_L$ induced by the symmetry of L is unitarily implementable and, *a fortiori*, there is no state ω which is θ-invariant. Thus, if the physically relevant states are Fock states with translationally invariant vacuum states, it is precluded that the symmetry is broken by evolution of any sort from a symmetric state to an asymmetric state since there are no symmetric states in the relevant sense.[10] Of course, this no-go result applies only to closed systems. It remains a possibility that in open systems there is some interesting sense of temporal symmetry breaking. But if that open system is a subsystem of a larger system that itself is a closed system, then that sense is inoperative for the larger system.

The features of spontaneous symmetry breaking that have been emphasized above cannot arise in the ordinary quantum mechanics of a system with a finite number of degrees of freedom since the Stone–von Neumann theorem says, roughly, that in such cases the representation of the canonical commutation relations is unique up to unitary equivalence. With $U_m(s) := \exp(iq_m s)$ and $V_n(t) := \exp(ip_n t)$, the Weyl form of the canonical commutation relations $p_n q_m - q_m p_n = -i\delta_{nm}$, etc., is given by $V_n(t)U_m(s) = U_m(s)V_n(t)\exp(ist\delta_{nm})$ and $U_m(s)U_n(t) - U_n(t)U_m(s) = 0 = V_m(s)V_n(t) - V_n(t)V_n(s)$. When the ranges of m and n, are finite, the Stone–von Neumann theorem says that the only irreducible representation of these relations by continuous unitary groups on Hilbert space is unitarily equivalent to the Schrödinger representation. It is the breakdown of the Stone–von Neumann theorem when the number of degrees of freedom is infinite – in particular, for the case of fields with an uncountable number of degrees of freedom – that opens up the possibility of spontaneous symmetry breaking. In algebraic terms, the Weyl relations generate a special C^*-algebra called (naturally) a Weyl algebra,[11] and this algebra admits innumerably many unitarily inequivalent representations. The existence of such representations was long known. The discovery of spontaneous symmetry breaking can be seen as the discovery of a particular physical mechanism

[10] It is sometimes said that non-Fock representations are needed to describe interacting quantum fields. This claim is disputable. And in any case the use of non-Fock representations renders inapplicable the above style of argument for showing that a symmetry of the Lagrangian is not unitarily implementable. Thus, if such representations are to host spontaneous symmetry breaking, a much different style of argument is needed.

[11] For an explicit construction of the Weyl algebra for the scalar Klein–Gordon field, see Wald (1994).

for generating such representations. Characterized in this way spontaneous symmetry breaking seems like pretty small potatoes. However, this impression is belied by the fact that an apparent problem with spontaneous symmetry breaking led to the Higgs mechanism, which is an essential ingredient in the Standard Model of elementary particles (see section 5, below).

4 Resolving the puzzles

We are now in a position to resolve the puzzles raised in section 2. The first puzzle has a quick *modus tollens* 'resolution': spontaneous symmetry breaking underscores the point that the vacuum in QFT is not a formless nothingness but rather a state with an intricate structure. A full resolution obviously requires an extended discussion of the structure of the vacuum, but it cannot be provided here.

The second puzzle is quickly dissolved. The degeneracy of the vacuum that occurs in spontaneous symmetry breaking does not contradict the usual statement about the uniqueness of the vacuum: the latter refers to the existence of a unique state vector satisfying various conditions within a given Fock space representation of the algebra of field operators, whereas the former refers to the existence of multiple unitarily inequivalent representations, each with its own unique vacuum state.

The resolution of the third puzzle requires a bit more commentary. Call a map $\mathcal{S} : \Phi \to \Phi'$ of the rays of a separable Hilbert space $\mathcal{H}$ a *Wigner* or *unbroken symmetry* just in case it preserves probabilities in the sense that for all rays Φ and Ψ, $|\langle\Phi|\Psi\rangle| = |\langle\Phi'|\Psi'\rangle|$ where $|\Phi\rangle$ and $|\Psi\rangle$ are vectors belonging to Φ and Ψ respectively. Wigner (1959) proved that $\mathcal{S}$ is unbroken iff there is a unitary or antiunitary operator $\hat{U}$ such that $|\Phi'\rangle = \hat{U}|\Phi\rangle$. An unbroken $\mathcal{S}$ induces a transformation of operators $\mathcal{T} : \hat{A} \to \hat{A}' = \hat{U}\hat{A}\hat{U}^{-1}$. Obviously, if $\hat{U}$ is unitary then $\mathcal{T}$ is an automorphism of $\mathcal{B}(\mathcal{H})$. Conversely, any automorphism of $\mathcal{B}(\mathcal{H})$ takes this form for a unitary $\hat{U}$ – in the terminology of section 3, $\mathcal{T}$ is an inner automorphism of the C^*-algebra $\mathcal{B}(\mathcal{H})$. How then can a symmetry in the guise of an automorphism of a C^*-algebra $\mathcal{A}$ fail to be an unbroken or Wigner symmetry? The answer is that two different senses of 'broken symmetry' are in play. For a broken symmetry in the sense of spontaneous symmetry breaking, the C^*-algebra $\mathcal{A}$ is not isomorphic to $\mathcal{B}(\mathcal{H})$; indeed, a representation π of $\mathcal{A}$ is into rather than onto $\mathcal{B}(\mathcal{H})$, and there is no continuous extension of $\pi(\mathcal{A})$ to all of $\mathcal{B}(\mathcal{H})$. An automorphism θ of $\mathcal{A}$ is broken in the sense of spontaneous symmetry breaking not because it is broken in the Wigner sense in that it fails to preserve probabilities but because it is not an automorphism of $\mathcal{B}(\mathcal{H})$. At the level of Hilbert space representations, the action of the broken θ is best construed as moving between unitarily inequivalent representations.

The fourth puzzle has been resolved in the sense that the existence of non-unitarily equivalent representations has been explained in terms of the algebraic apparatus.

It remains to give more of a feeling for what these representations mean in terms of physics. If one tries to get a grasp on the meaning of unitarily inequivalent representations by insisting on thinking of the vacuum states from different representations as all belonging to one big Hilbert space, then the different vacuum states must lie in different superselection subspaces in that a genuine superposition of states from two different such subspaces is not physically meaningful.[12] This follows from the fact that for pure states ω_1 and ω_2 on a C^*-algebra $\mathcal{A}$, unitary inequivalence is the same as *disjointness*, the latter of which means intuitively that any vector from the GNS representation of ω_1 is 'orthogonal' to any vector in the GNS representation of ω_2 and vice versa.[13] In ordinary QM or QFT one can make a symmetric state by superposing asymmetric states, and 'measurement collapse' of a superposition onto one of its 'branches' can produce an asymmetric state from a symmetric one. But neither of these things can happen if an 'asymmetric state' is understood to mean not just that the state is not invariant under the relevant symmetry but also that the symmetry is not unitarily implementable with respect to the state.

5 Issues of interpretation

The above rough guide to spontaneous symmetry breaking provides an entry point into the subject without revealing the subleties and unresolved issues that a more thorough guide would unveil. Here I will simply mention three important issues that the more assiduous explorers will encounter, leaving it to them to investigate the details and to provide their own resolutions.

In section 3 I indicated that spontaneous symmetry breaking arises in cases of continuous symmetries falling under Noether's first theorem: if the symmetry were unitarily implementable, it would be generated by the global charge operator $\hat{Q}$, which is obtained by integrating over all space the time component $\hat{j}^0$ of the current operator corresponding to the conserved Noether current; but, given plausible assumptions such as the translation invariance of the vacuum, provably there is no such self-adjoint operator. Actually, the charge operator $\hat{Q}_V$ corresponding to a finite volume V of space is well defined. It is only the infinite volume limit $\hat{Q} = \lim_{V\to\infty} \hat{Q}_V$ that is ill defined. Since actual physical systems exhibiting spontaneous symmetry breaking – e.g. ferromagnets and superconductors – occupy only a finite volume, it needs to be asked whether or not the distinctive features of spontaneous symmetry breaking discussed above are merely artifacts of the infinite volume idealization.

[12] And in the case of the above toy example where the cardinality of the degenerate vacuum states is that of the continuum, the Hilbert space won't be separable.

[13] For a precise definition of disjointness, see Bratteli and Robinson (1987, pp. 370–1). As emphasized by Castellani (this volume, Part III), the existence of a superselection rule for ground states holds not only for QFT but more generally for quantum systems with an infinite number of degrees of freedom.

The second issue arises from Goldstone's theorem (see Goldstone *et al.*, 1962) which says, roughly, that in cases of spontaneous symmetry breaking arising from a finite parameter Lie group symmetry, the imposition of standard quantum field theoretic assumptions, such as Poincaré invariance and locality, implies the existence of massless scalar bosons. Since all of the experimental evidence indicates that such particles do not exist, the upshot is a dilemma: either spontaneous symmetry breaking does not occur or else the standard quantum field theoretic assumptions must go.

The response that most particle theorists have settled on is known as the *Higgs mechanism*. It passes through the horns of the dilemma by changing the problem. Additional fields are added in such a fashion that the new Lagrangian is invariant under an infinite-dimensional Lie group, where the group parameters are arbitrary functions of the independent variables. The situation now falls under Noether's second theorem, and the symmetries at issue are gauge symmetries. It is found that the gauge can be set in such a way that the Goldstone bosons are absent, which shows that in the changed environment these particles can be dismissed as gauge epiphenomena.

The Higgs mechanism has been incorporated into the Standard Model of particle physics not only because it suppresses Goldstone bosons but also because it provides a way to give the particles their masses. But because the new theory is a gauge theory, the upshot for the nature of spontaneous symmetry breaking remains opaque until the veil of gauge is removed and the theory is reduced to its gauge-independent content.[14] Suppose that this content is described by local fields which, when quantized, obey the standard assumptions of QFT, such as Poincaré invariance and locality – otherwise the implementation of the Higgs mechanism would necessitate a radical revision of QFT. On pain of resurrecting Goldstone bosons, the reduced theory cannot admit a finite parameter Lie group as a symmetry group. Goldstone's theorem does not apply if the reduced theory admits a discrete symmetry group, but neither does Noether's first theorem nor the argument sketched in the preceding section for the spontaneous breakdown of symmetry. There are heuristic arguments for the spontaneous breakdown of discrete symmetries in QFT, but as far as I am aware there are no demonstrations of even modest rigour for such a breakdown. In sum, the fate of spontaneous symmetry breaking in the Higgs model is unsettled.

[14] As discussed elsewhere in this volume, there is in principle a method for accomplishing this goal. If the variational symmetries consist of transformations that depend on arbitrary functions of the independent variables, then the Hamiltonian formulation contains constraints. The subclass of first-class constraints generates the gauge transformations on the Hamiltonian phase space. If various technical obstructions do not intervene, quotienting out the gauge orbits yields the reduced phase space whose canonical coordinates are gauge-invariant quantities.

The third issue also revolves around the gauge concept. In the case of the Higgs mechanism the concept of gauge is invoked in order to overcome an apparent failure of determinism that results when the action admits a variational symmetry group of transformations that depend on arbitrary functions of the independent variables. But apart from considerations of determinism, there can be other grounds for seeing gauge freedom at work. For example, there might be reasons for thinking that a complex-valued scalar field φ is not observable, whereas combinations such as $\varphi^*\varphi$ are and, consequently, that the transformations $\varphi \to \varphi' = \exp(i\alpha)$, $\partial_\mu \alpha = 0$, are gauge transformations in the sense that they connect different descriptions of the same physical state. While such a line is defensible, it leads to peculiar consequences in cases that would normally be described as cases of spontaneous symmetry breaking. In the first place, assuming that $\varphi \to \varphi' = \exp(i\alpha)$ is a symmetry of the Lagrangian, the stance under discussion implies that the symmetry isn't broken in the relevant sense: the world doesn't break the symmetry by choosing one from among an infinity of physically distinct states; it is rather that we make a conventional choice from among different gauges describing the same physical situation. In the second place, since different values of α label unitarily inequivalent representations, it follows from the stance under discussion that these different representations must be treated as physically equivalent. This is not an unknown position. For example, some of the pioneers of algebraic QFT promoted the concept of *weak equivalence* (see appendix) of representations as the explication of physical equivalence, and they proved that all faithful representations of a C^*-algebra are weakly equivalent. However, there are reasons to doubt that weak equivalence is sufficient for physical equivalence (see Arageorgis *et al.*, 2002) and, thus, reasons to doubt the current line on gauge.

6 Conclusion

In addition to its implications for the topic of laws and symmetries, spontaneous symmetry breaking is a marvellous vehicle for probing some of the deepest and most important problems in the foundations of QFT. But by the same token, since these problems are notoriously difficult and contentious, they collectively provide a barrier to a full understanding of the basis and ramifications of spontaneous symmetry breaking. This is, perhaps, a large part of the reason that there is little of substance in the philosophical literature about spontaneous symmetry breaking in QFT. If this rough guide to spontaneous symmetry breaking serves to encourage philosophers of science to explore the topic, it will have fulfilled part of its purpose. The other part will have been fulfilled if it serves to indicate to physicists how the algebraic formulation of QFT, though useless for calculations, helps to clarify foundational issues.

Appendix

A C^**-algebra* $\mathcal{A}$ is an algebra, over the field $\mathbb{C}$ of complex numbers, with an involution $*$ satisfying: $(A^*)^* = A$, $(A+B)^* = A^* + B^*$, $(\lambda A)^* = \bar{\lambda} A^*$ and $(AB)^* = B^* A^*$ for all $A, B \in \mathcal{A}$ and all complex λ (where the overbar denotes the complex conjugate). In addition, a C^*-algebra is equipped with a norm, satisfying $\|A^* A\| = \|A\|^2$ and $\|AB\| \leq \|A\| \, \|B\|$ for all $A, B \in \mathcal{A}$, and is complete in the topology induced by that norm. It is assumed here that $\mathcal{A}$ contains a unit 1 such that $1A = A1 = A$ for all $A \in \mathcal{A}$. Observables are identified with self-adjoint elements of $\mathcal{A}$, i.e. elements A such that $A^* = A$. A *state* on $\mathcal{A}$ is a linear functional ω that is normed ($\omega(1) = 1$)) and positive ($\omega(A^* A) \geq 0$ for all $A \in \mathcal{A}$). ω is a *pure state* iff it cannot be written as $\lambda_1 \omega_1 + \lambda_2 \omega_2$ where $\omega_1 \neq \omega_2$ and $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$.

A *representation* of a C^*-algebra $\mathcal{A}$ is a mapping $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ from the abstract algebra into the concrete algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ such that $\pi(\lambda A + \mu B) = \lambda \pi(A) + \mu \pi(B)$, $\pi(AB) = \pi(A)\pi(B)$, and $\pi(A^*) = \pi(A)^\dagger$ for all $A, B \in \mathcal{A}$ and all $\lambda, \mu \in \mathbb{C}$. A representation is *faithful* if $\pi(A) = 0$ implies $A = 0$. A fundamental theorem due to Gelfand, Naimark, and Segal (GNS) guarantees that for any state ω on $\mathcal{A}$ there is a representation $(\pi_\omega, \mathcal{H}_\omega)$ of $\mathcal{A}$ and a cyclic vector $|\Psi_\omega\rangle \in \mathcal{H}_\omega$ (i.e. $\pi_\omega(\mathcal{A})|\Psi_\omega\rangle$ is dense in $\mathcal{H}_\omega$) such that $\omega(A) = \langle \Psi_\omega | \pi_\omega(A) | \Psi_\omega \rangle$ for all $A \in \mathcal{A}$; moreover, this representation is the unique, up to unitary equivalence, cyclic representation. Two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of a C^*-algebra $\mathcal{A}$ are said to be *unitarily equivalent* just in case there is an isomorphism $\hat{U} : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\hat{U} \pi_1(A) \hat{U}^{-1} = \pi_2(A)$ for all $A \in \mathcal{A}$.

The *kernel* $Ker(\pi)$ of a representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ is defined as $\{A \in \mathcal{A} : \pi(A) = 0\}$. Two representations are said to be *weakly equivalent* iff their kernels are the same. Unitary equivalence of representations implies weak equivalence, but not conversely. Obviously all faithful representations of a given C^*-algebra are weakly equivalent. It turns out that two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathcal{A}$ are weakly equivalent iff for any algebraic state ω_1 corresponding to a density matrix on $\mathcal{H}_1$ and any $A_1, \ldots A_n \in \mathcal{A}$ and any $\varepsilon_1, \ldots, \varepsilon_n > 0$, there exists a state ω_2 corresponding to a density matrix on $\mathcal{H}_2$ such that for all $i = 1, \ldots, n$, $|\omega_1(A_i) - \omega_2(A_i)| < \varepsilon_i$. This finite operational equivalence is arguably not sufficient for full physical equivalence.

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