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INTERPRETING PROBABILITIES IN QUANTUM FIELD THEORY AND QUANTUM STATISTICAL MECHANICS

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1 Introduction

In ordinary nonrelativistic quantum mechanics (QM), the observables pertaining to a system typically form the self-adjoint part of the algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators acting on a Hilbert space $\mathcal{H}$.[1] $\mathfrak{B}(\mathcal{H})$ is an algebra of the genus *von Neumann* and the species *Type I factor*.[2] Here we consider quantum systems whose observable-algebras belong to the same genus, but correspond to more exotic species. Settings in which the exotic species occur include relativistic quantum field theory (QFT) and the thermodynamic limit of quantum statistical mechanics (QSM), reached by letting the number of systems one considers and the volume they occupy go to infinity while keeping their density finite. The aim of this essay is to articulate the impact the non-Type-I von Neumann algebras have on the interpretation of quantum probability.

We proceed as follows. Section 2 sets the stage for the rest of our discussion by highlighting key elements of the formalism and interpretation of quantum probability in the familiar setting of Type-I von Neumann algebras. Key elements of the formalism include Gleason's Theorem and Lüders' Rule of Conditionalization; key elements of its standard interpretation include the use of minimal projection operators to characterize not only the preparation of quantum states but also the results of quantum measurements, as well as the manner in which the former assign probabilities to the latter. Section 3 motivates the significance of non-Type-I algebras by describing some physical situations that give rise to them. It also reviews some of the novel features of these algebras, including features that might seem to be impediments to

[1] For the sake of simplicity we will assume, unless otherwise noted, that $\mathcal{H}$ is a separable Hilbert space.

[2] When superselection rules are present and the superselection rules commute, the relevant von Neumann algebra is a Type-I non-factor of the form $\bigoplus_j \mathfrak{B}(\mathcal{H}_j)$ acting on $\bigoplus_j \mathcal{H}_j$. We will ignore this complication here. For an overview of superselection rules and their implications for the foundations of quantum theory, see Earman 2008b.

interpreting quantum probability in the general setting. Section 4 examines this appearance. It allays the worry that key components of the *formalism* of quantum probabilities familiar from ordinary QM are artifacts of features peculiar to Type-I factor von Neumann algebras. In particular, it is shown how to generalize Gleason's Theorem and Lüders' Rule of Conditionalization to von Neumann algebras of arbitrary type. Section 5 identifies some reasons to think that *interpretations* of quantum probabilities familiar from ordinary QM do not extend, without significant adaptation, to more general settings, and discusses some possible escapes from the difficulties engendered by non-Type-I algebras. Conclusions are presented in Sec. 6. The appendices review the basics of operator algebras, the classification of von Neumann algebras, and lattice theory.

This introductory section closes with a brief sketch of some of the difficulties encountered with non-Type-I algebras. Many familiar interpretations of ordinary QM exploit a feature of $\mathfrak{B}(\mathcal{H})$ that an arbitrary von Neumann algebra $\mathfrak{M}$ may not necessarily share. $\mathfrak{B}(\mathcal{H})$ has *atoms*, which are *minimal* projection operators. (Intuitively, a projection operator E is minimal in a von Neumann algebra $\mathfrak{M}$ if $\mathfrak{M}$ contains no nontrivial projection whose range is a proper subspace of E's range; see App. B for details.) A tactic widespread in the interpretation of ordinary QM is to use atoms in $\mathfrak{B}(\mathcal{H})$ to interpret quantum probabilities. In the orthodox collapse interpretation, atoms code the endpoints of a measurement collapse. Atoms code the determinate 'value states' recognized by modal interpretations and the 'relative states' purveyed by many-worlds interpretations. In these interpretations, atoms characterize the situations that are assigned probabilities by a density-operator state W on $\mathfrak{B}(\mathcal{H})$, as well as explain why the Born Rule accurately predicts the values of those probabilities. For example, supposing W to be nondegenerate, a stock modal interpretation puts possible value states of a system described by W in one-to-one correspondence with the projection operators furnishing W's spectral resolution, which are atoms. The modal interpretation explicates the Born Rule by setting the probability that the system occupies the value state coded by the atom E equal to $\mathrm{Tr}(WE)$.

We do not contend that in the Type-I case the interpretation of quantum probability *must* be mediated by atoms—only that it typically is. Part of the point of considering more general cases is to determine whether what is typical is nevertheless dispensable—to determine, that is, whether and how the interpretation of quantum probability might proceed in the absence of features to which interpreters habitually appeal. Section 5 will suggest that although there are candidates formally well qualified as atoms to play a role in the interpretation of probabilities assigned by states on a von Neumann algebra $\mathfrak{M}$ that does not contain atoms, they lack the metaphysical and practical qualifications atoms enjoy. In the presence of exotic von Neumann algebras, quantum probability theory is a formalism still in search of an interpretation.[3]

2 Ordinary QM: Formalism and interpretation

2.1 *Formalism: Gleason's Theorem and Lüders' Rule*

Much of our discussion will focus on *normal states* on a von Neumann algebra $\mathfrak{M}$ acting on a Hilbert space $\mathcal{H}$. These are the states that are generated by density operators (positive trace class operators) on $\mathcal{H}$; that is, ω is a normal state for $\mathfrak{M}$ just in case there is a density operator W such that $\omega(A) := \mathrm{Tr}(WA)$ for all $A \in \mathfrak{M}$. There are good mathematical reasons for focusing on such states; for example, countable additivity is a mathematically desirable feature, and this feature obtains for all and only the normal states.[4] But physical considerations also motivate the desire for normality. There is a folklore in the physics literature to the effect that states that are not normal with respect to local observable-algebras (e. g. algebras associated with double-diamond regions in Minkowski space-time) are not physically realizable in QFT because an infinite energy would be required to prepare them.

Segal (1959) suggested another reason to restrict attention to normal states. Physics aims to articulate physical laws, which we might understand as interrelations between physical magnitudes, or their values, that every physical state instantiates. Having identified a von Neumann algebra $\mathfrak{M}$ as the algebra of physical magnitudes, this understanding of physical law underwrites a characterization of physical states: the physical states on $\mathfrak{M}$ are those that respect the law-like relationships between magnitudes in $\mathfrak{M}$. Some magnitudes in $\mathfrak{M}$ are limits in the weak operator topology of sequences of other magnitudes in $\mathfrak{M}$. Let A_i be a sequence of elements of $\mathfrak{M}$. A state ω on $\mathfrak{M}$ is *weakly continuous* only if it follows from the fact that an element A of $\mathfrak{M}$ is a limit in the weak operator topology of a sequence A_i of elements of $\mathfrak{M}$ that $\lim_{i\to\infty} \omega(A_i) = \omega(A)$. A state ω which is not weakly continuous can assign the weak limit of a sequence of elements of $\mathfrak{M}$ a value that is not determined by the values ω assigns members of the sequence. This could hinder ω from instantiating laws whose expression requires the taking of a weak limit. Arguably, the integral form of Schrödinger's Equation is such a law. The one-parameter group of unitarities $U(t)$ implementing the time evolution of a quantum system has a self-adjoint infinitesimal generator—the observable that Schrödinger identifies as the Hamiltonian H of a system—only if

[3]There are many good reviews of quantum probability, e. g. Hamhalter 2003, Rédei & Summers 2007, and Streater 2000. Our goal here is to emphasize interpretation problems that have not found their way into philosophical consciousness.

[4]For a von Neumann algebra acting on a nonseparable $\mathcal{H}$, normality is equivalent to complete additivity; but for a separable $\mathcal{H}$, countable additivity suffices for normality.

$U(t+\delta)$ converges in the weak operator topology to $U(t)$ as $\delta \to 0$. In this case, H's spectral projections are strong (and therefore weak) operator limits of polynomials p_i of $U(t)$ (see Prugovečki 1981, pp. 335–9). A state ω that fails to be weakly continuous can assign H an expectation value different from the limit of the expectation values it assigns the polynomials p_i. Such an expectation value assignment fails to respect the functional and limiting relationships Schrödinger's Law posits between the family of evolution operators $U(t)$ and a Hamiltonian generator of that family. In this sense, a state that fails to be weakly continuous can fail to instantiate Schrödinger's Law of time development.

In general, states that fail to be weakly continuous threaten to upset nomic relations $\mathfrak{M}$ makes available. To avert this threat, one might reject as unphysical states on $\mathfrak{M}$ that fail to be weakly continuous. This rules out non-normal states because non-normal states are not weakly continuous (see Bratteli & Robinson 1987, Thm 2.4.21, p. 76, and the definitions that precede it). Together these considerations help to explain why non-normal states are often referred to in the literature as 'singular' states.

Since the present concern is with ordinary QM, specialize for the moment to the case of $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ with $\mathcal{H}$ separable, and let $\mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big)$ be the projection lattice of $\mathfrak{B}(\mathcal{H})$.[5] Obviously a normal state on $\mathfrak{B}(\mathcal{H})$ defines a countably additive measure on $\mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big)$. The converse is the assertion that for any countably additive measure $\mu\colon \mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big) \to [0,1]$ there is a unique normal state ω on $\mathfrak{B}(\mathcal{H})$ such that $\omega(E) = \mu(E)$ for all $E \in \mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big)$ (or, equivalently, there is a density operator W such that $\mathrm{Tr}(WE) = \mu(E)$ for all $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$). This assertion is not true in all generality; in particular, it fails if $\dim(\mathcal{H}) \leq 2$. But a highly nontrivial result, known as Gleason's Theorem, shows that the assertion is true for $\dim(\mathcal{H}) > 2$.[6]

Another (relatively shallow but) important result provides the motivation for Lüders' Rule. Let $\mu\colon \mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big) \to [0,1]$ be a countably additive measure. By Gleason's Theorem, if $\dim(\mathcal{H}) > 2$ there is a unique normal state ω on $\mathfrak{B}(\mathcal{H})$ that extends μ to $\mathfrak{B}(\mathcal{H})$. Let $E \in \mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big)$ be such that $\mu(E) \neq 0$. Then ω_E, given by

$$\omega_E(A) := \frac{\omega(EAE)}{\omega(E)} \qquad \text{for all } A \in \mathfrak{B}(\mathcal{H}),$$

is a normal state. This state has the following property:

(L) *For any* $F \in \mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big)$ *such that* $F \leq E$, $\omega_E(F) = \mu(F)/\mu(E)$,

[5] See App. C for lattices.

[6] In the philosophy literature, Gleason's Theorem is often stated as a result about probability measures on the lattice of closed subspaces of $\mathcal{H}$. There is a natural isomorphism of this lattice and $\mathcal{P}\big(\mathfrak{B}(\mathcal{H})\big)$.

where $\leq$ is the relation of subspace inclusion. Furthermore, ω_E is the unique normal state with the property (L) (see Bub 1977). Using the trace prescription, the state ω_E can be rewritten as

$$\omega_E(A) = \frac{\mathrm{Tr}(EAEW)}{\mathrm{Tr}(EW)} \qquad \text{for all } A \in \mathfrak{B}(\mathcal{H}),$$

where W is the density operator corresponding to the normal state ω.

These properties of ω_E are taken to motivate its interpretation as giving a conditional quantum probability. This rule for quantum conditionalization is commonly referred to as 'Lüders' Rule' (see Bub 1977 and Hughes 1989, Sec. 8.2). In the above form, it might be used to assign statistics to experiments subsequent to *selective* measurements, that is, measurements after which the experimenter retains only outcomes corresponding to the condition E. A form of Lüders' Rule applicable to nonselective measurements will be discussed in Sec. 4.

Allied with Lüders' Rule is the important notion of a *filter,* which we will state for von Neumann algebras in general. Intuitively, a filter for a state φ is a projection operator E_φ such that Lüders-conditionalizing an arbitrary state ω on that projection operator yields a state ω_{E_φ} identical to φ, *provided* φ *and* ω *aren't 'orthogonal' to begin with.* (If the proviso fails, there's no way to Lüders-conditionalize ω to obtain φ.) Officially, a filter is defined as follows. Let φ be a normal state on a von Neumann algebra $\mathfrak{M}$ and let S_φ be its support projection. (The *support projection* or *carrier* of a state φ on a von Neumann algebra $\mathfrak{M}$ is the smallest projection in $\mathfrak{M}$ to which φ assigns a probability of 1. If it exists, the support projection is unique.) The projection $E_\varphi \in \mathfrak{M}$ is a *filter* for φ just in case, for any normal state ω on $\mathfrak{M}$, if $\omega(S_\varphi) \neq 0$ then[7]

$$\omega_{E_\varphi}(A) := \frac{\omega(E_\varphi A E_\varphi)}{\omega(E_\varphi)} = \varphi(A) \qquad \text{for all } A \in \mathfrak{M}.$$

$\omega_{E_\varphi}(A)$ should be recognizable both as the state ω Lüders-conditionalized on a *Yes* outcome of an E_φ-measurement, and as the post-measurement state of a system initially in ω and subject to such a measurement, *according to the Projection Postulate.* In this *interpretation-dependent* sense, we can directly observe and prepare φ.

The following fact, whose proof is sketched at the end of this section, will be important in what follows:

Fact 1 (Purification) *A normal state* φ *on a von Neumann algebra* $\mathfrak{M}$ *has a filter if and only if it is pure.*

[7] $\omega(S_\varphi) \neq 0$ is an expression of a requirement that may be more intuitively phrased as follows: ω does not assign the state φ probability 0.

In the case of ordinary QM, i.e. $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$, the normal pure states coincide with the vector states (see App. A), and all of these states have filters. For Type-III algebras, which will feature in what follows, the normal states also coincide with the vector states; but there are no normal pure states and, hence, no normal state on such an algebra has a filter. This raises concerns about preparation procedures which will be addressed in due course.

Proof *Pure ⇒ filter:* if φ is a normal pure state on $\mathfrak{M}$ then its support projection S_φ is a filter for φ.

Since the normal state φ is pure, its support projection S_φ is an atom (cf. p. 264) of $\mathfrak{M}$.[8] Therefore, $S_\varphi \mathfrak{M} S_\varphi$ consists of scalar multiples of S_φ (Kadison & Ringrose 1997b, Prop. 6.4.3). Thus, for $A \in \mathfrak{M}$, $S_\varphi A S_\varphi = c_A S_\varphi$ (where the constant c_A may depend on A). So $\varphi(S_\varphi A S_\varphi) = \varphi(A) = \varphi(c_A S_\varphi) = c_A \varphi(S_\varphi) = c_A$ (since $\varphi(S_\varphi) = 1$). For any state ω, $\omega(S_\varphi A S_\varphi) = \omega(c_A S_\varphi) = \omega\big(\varphi(A)\, S_\varphi\big) = \varphi(A)\,\omega(S_\varphi)$. If ω is normal and if $\omega(S_\varphi) \neq 0$, then Lüders-conditionalizing ω on S_φ yields a state ω_{S_φ} whose action on $\mathfrak{M}$ will coincide with φ's:

$$\omega_{S_\varphi}(A) \;:=\; \frac{\omega(S_\varphi A S_\varphi)}{\omega(S_\varphi)} \;=\; \varphi(A) \qquad \text{for all } A \in \mathfrak{M}.$$

This establishes that S_φ satisfies the definition of a filter E_φ for φ.

Filter ⇒ pure: if φ is a mixed (nonpure) normal state on $\mathfrak{M}$ then it has no filter.

Suppose for *reductio* that φ has a filter E_φ. Let S_φ be φ's support projection. If ω is a normal state such that $\omega(S_\varphi) \neq 0$, then (according to the definition of a filter) $\varphi(E_\varphi) = \omega(E_\varphi E_\varphi E_\varphi)/\omega(E_\varphi) = 1$. Since $\varphi(E_\varphi) = 1$, $S_\varphi \leq E_\varphi$. Because φ is mixed, S_φ is not minimal in $\mathfrak{M}$. So there is some projection $F \in \mathfrak{M}$ such that $F < S_\varphi$, which implies that $F < E_\varphi$. (Here '$<$' is proper subspace inclusion.) Now let ψ be a normal state on $\mathfrak{M}$ with support projection F. Because E_φ is a filter for φ, $\psi(E_\varphi F E_\varphi)/\psi(E_\varphi) = \varphi(F)$. Since $F < E_\varphi$ and $\psi(F) = 1$, the l.h.s. becomes $\psi(F)/\psi(E_\varphi) = 1$. But the r.h.s. is less than 1, because S_φ is the smallest projection φ maps to 1 and $F < S_\varphi$. We have our contradiction and can conclude that no mixed normal state has a filter. □

2.2 *Interpreting probabilities in ordinary* QM

With the mention of the Projection Postulate, we have strayed into the territory of interpretation. We can resolve the question of the interpretation of quantum probability into two parts. *First*, how do quantum systems come to occupy conditions—call them *quantum states*—that assign specifiable (and ergo testable)

[8]This follows from the fact that a state ω on a C*-algebra $\mathfrak{A}$ is pure if and only if it's the only state that vanishes on its left kernel, the set of positive elements of $\mathfrak{A}$ mapped to 0 by ω (Kadison & Ringrose 1997b, Thm 10.2.7). For the argument detailed, see Clifton & Halvorson 2001.

probabilities? *Second*, how ought one to understand the probabilities these quantum states assign?

These parts overlap with the traditional problems of preparation and measurement. As befits its larger profile, the latter contains a *set* of questions, some of which we've already alluded to. (Q1) What are the *recipients* of quantum probability assignments? Put another way, this is recognizably the central question of quantum interpretation: what configurations are possible for a system in a quantum state, and what value-attributing propositions determinately true or false of it? We will call this the *problem of characterization*. (Q2) Why is the Born Rule an empirically adequate predictor of the values of these probabilities? We will call this the *problem of explication*. (Q3) What is the nature—epistemic, non-epistemic, decision-theoretic, or what have you—of these probabilities? We won't give this problem a special name, because it arises whenever a physical theory assigns probabilities, but we will note that it would be elegant if quantum probabilities could be brought to heel by existing strategies for the interpretation of probability. In this essay, we don't take a stand on how to answer (Q3). (Fortunately, other contributors to this volume are less pusillanimous.) We do try to delineate complications that QFT and the thermodynamic limit of QSM present for standard strategies for interpreting the probabilities of ordinary QM.

By examining the account of quantum probabilities urged on students of quantum mechanics by their textbooks, we can see how elements of the quantum probability formalism circumscribe and facilitate the interpretation of quantum probabilities. The crux of the textbook account is the putative phenomenon of measurement collapse, in which the measurement of an observable A (assumed to have a purely discrete spectrum) on an object in the superposition $|\psi\rangle = \sum_i c_i |\alpha_i\rangle$ of A-eigenstates $|\alpha_i\rangle$ *collapses* that superposition to *one* of those eigenstates, with $|c_n|^2$ giving the probability of a collapse to the eigenstate $|\alpha_n\rangle$. The outcome of a measurement suffering such a collapse is the A-eigenvalue associated with $|\alpha_n\rangle$. In the parlance of the previous section, the A-eigenprojection E_n^A whose range is the subspace spanned by $|\alpha_n\rangle$ is a *filter* for $|\alpha_n\rangle$. This enables a general statement of the textbook account of state preparation: a measurement of the filter E_φ yielding the outcome *Yes* prepares a quantum system in the state φ. By Fact 1, a state φ susceptible to such preparation is a pure (= vector) state on $\mathfrak{B}(\mathcal{H})$. This textbook account of state preparation not only covers the preparation of quantum systems in known states, when those states possess filters, but also justifies calculating probabilities of measurement outcomes subsequent to such preparation using Lüders' Rule.

When the prepared state $|\alpha_n\rangle$ is subject to a B-measurement, the textbook account again invokes measurement collapse. Assume that B is nondegenerate. (This assumption isn't essential to the orthodox account, but it streamlines its presentation. We will lift it presently.) Expressed as a superposition of

B-eigenstates, $|\alpha_n\rangle = \sum_i d_i |\beta_i\rangle$. Recipients of quantum probabilities assigned by $|\alpha_n\rangle$ are the possible outcomes of the B-measurement, each of which is coded by a unique B-eigenstate, associated with an atom E_i^B in the projection lattice of $\mathfrak{B}(\mathcal{H})$. What's true of the outcome coded by $|\beta_i\rangle$ is determined by applying the eigenvector–eigenvalue link to $|\beta_i\rangle$. Thus the orthodox account solves the characterization problem. It solves the explication problem by assigning the standard Born Rule probability ($|d_i|^2$) to a collapse to the outcome coded by $|\beta_i\rangle$.

Lüders' Rule characterizes the state prepared by measuring a filter, the state whose assignment of quantum probabilities the orthodox account characterizes and explicates. Lüders' Rule supposes the system to occupy a normal state prior to the filter measurement. The pre-preparation states amenable to the textbook account are exactly those characterized by Gleason's Theorem.

But since the textbook account and the Projection Postulate generate well-known difficulties it would be desirable to have a collapse-free reading of quantum probabilities. Here is one such reading.[9] Start with the question of how quantum systems come to occupy states that assign specifiable probabilities. If ω is the pre-measurement state and a measurement of the projection E returns a *Yes* answer, then the Projection Postulate asserts that the Lüders-conditionalized state ω_E is the post-measurement state. The no-collapse reading denies this. On that reading, ω_E is not literally the post-measurement state, wrenched from ω by the agency of measurement collapse. Schrödinger evolution is exceptionless. Nevertheless, ω_E summarizes a set of probabilities *conditional on E* implicit in ω. Because measurements subsequent to the E-measurement occur in the scope of the condition, ω_E is the appropriate predictive instrumentality to use when considering those measurements—appropriate because Lüders' Rule is the correct expression for quantum conditional probabilities, not because collapse has occurred. So the strategy of the Projection Postulate–free reading is to relieve ourselves of measurement collapse by taking the Lüders Rule to be basic, a move motivated by the capacity of Lüders' Rule, taken as basic, to save the very phenomena the Projection Postulate was taken to explain.[10]

As for a collapse-free account of the probabilities *assigned* by a normal state corresponding to a density operator W on $\mathfrak{B}(\mathcal{H})$, the options are legion. Faced with the measurement problem—the problem that Schrödinger's cat winds up, like a radioactive atom but unlike any feline every encountered on earth, superposed between life and death—collapse interpretations court the miraculous by interrupting Schrödinger's Law–governed unitary dynamics by measurement collapse. The family of modal interpretations is united by the conviction that the solution to the measurement problem consists not in novel dynamics, but in

[9] For a development, see van Fraassen 1991.

[10] Ruetsche (2003) extends the strategy to a Projection Postulate–free account of preparation.

novel ways of conceptualizing superposition. Thus they seek to understand the cat's superposed state as consistent with its being determinately alive or determinately dead. We will focus on a modal interpretation according to which each projection in W's spectral resolution codes a value state of a system described by a density-operator state W, with the eigenvector–eigenvalue link providing the decoder. This is the modal solution of the characterization problem. The modal interpretation solves the explication problem with the help of some additional assumptions. Let W describe the reduced state of an apparatus after a measurement that perfectly correlates the eigenstates $|p_i\rangle$ of the pointer observable P with eigenstates $|o_i\rangle$ of the object observable being measured, and let the pre-measurement state of the object system be $\sum_i c_i |o_i\rangle$. Then $W = \sum_i |c_i|^2 E_i^P$. According to the modal interpretation, the probability that the apparatus enjoys the value state coded by E_n^P is $|c_n|^2$, which is exactly the Born Rule probability the pre-measurement object state assigns the n^{th} outcome. Other no-collapse interpretations can be shoe-horned, with more or less force, into the modal mold (see Sec. 5.2.3). Notice how Gleason's Theorem underwrites the generality of the modal interpretation: geared to the interpretation of probabilities assigned by density-operator states, the modal interpretation is therefore geared to the interpretation of *any* normal state on *any* $\mathfrak{B}(\mathcal{H})$, provided $\dim(\mathcal{H}) > 2$.

What happens to the modal interpretation if W is degenerate? There is disagreement. One option is to allot W as many distinct value states as it has distinct eigenvalues, with each value state coded by a (possibly nonatomic) eigenprojection. In the limiting case that W is a tracial state (e. g. proportional to the identity operator on $\mathcal{H}$; this can occur only when $\mathcal{H}$ is finite-dimensional), this option has the consequence that the only propositions true of a system in state W are those attributing values to multiples of the identity operator—a highly uninformative solution to the characterization problem. In this case, the modal explication of Born Rule probabilities is similarly banal: the only probabilities explicated are certainties assigned to tautologies. Another option, more informative but potentially less principled, is to pick from among degenerate W's myriad eigenbases a privileged one, corresponding to a complete set of orthogonal atoms, and to use these to characterize value states and explicate Born Rule probabilities. Similar points apply to the collapse interpretation when the measured observable A is degenerate. Let E_n^A be an A-eigenprojection associated with a degenerate eigenvalue. The Projection Postulate asserts a system in the state ω and subject to an A-measurement yielding this eigenvalue to occupy the state $\omega_{E_n^A}$, a mixed state. One might suspect that there are more truths to be had about such a system than can be obtained by applying the eigenvector–eigenvalue link to $\omega_{E_n^A}$. The Maximal-Beable Approach, detailed in Sec. 5, gives voice to this suspicion.

This cursory discussion suggests that prevailing practices of probability interpretation in ordinary QM take the set of states assigning quantum probabilities to be the set described by Gleason's Theorem, implicate Lüders' Rule (either as a presupposition, as in no-collapse approaches, or a consequence, as in collapse approaches) in the account of how probability-assigning states are prepared, and use atoms in $\mathfrak{B}(\mathcal{H})$ to characterize the recipients of quantum probabilities as well as to explicate their obedience to the Born Rule.

3 Non-Type-I algebras

3.1 *Relevance to physics of non-Type-I algebras*

Von Neumann algebras not isomorphic to $\mathfrak{B}(\mathcal{H})$ for some separable $\mathcal{H}$ are not artefacts of a mathematical excess. Indeed, they are typical of quantum theories of systems with infinitely many degrees of freedom. Here are some examples.

In the algebraic formulation of relativistic QFT (see Haag 1996) a C^*-algebra $\mathcal{A}(\mathcal{O})$ of observables is associated with each open bounded region of $\mathcal{O} \subset \mathcal{M}$ of Minkowski space-time $\mathcal{M}$. This association is assumed to have the net property that if $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$. The quasi-local algebra $\mathcal{A}(\mathcal{M})$ for the entirety of Minkowski space-time is given by $\overline{\bigcup_{\mathcal{O}\in\mathcal{M}} \mathcal{A}(\mathcal{O})}$, where the overbar denotes the closure with respect to the C^*-norm. The von Neumann algebra $\mathfrak{M}(\mathcal{O})$ affiliated with a C^*-algebra $\mathcal{A}(\mathcal{O})$ depends on the representation π of $\mathcal{A}(\mathcal{O})$; specifically $\mathfrak{M}(\mathcal{O}) := \pi\big(\mathcal{A}(\mathcal{O})\big)''$, where '$''$' denotes the double commutant.[11] Similar remarks apply to the von Neumann algebra $\mathfrak{M}(\mathcal{M})$ affiliated with the quasi-local algebra $\mathcal{A}(\mathcal{M})$. Typically the physically relevant representation is taken to be the Gelfand–Neimark–Segal (GNS for short) representation picked out by some distinguished state, e.g. the vacuum state. For the Minkowski vacuum state for the mass $m \geq 0$ Klein–Gordon field, if $\mathcal{O}$ is a region with non-empty space-like complement, the standard axioms for algebraic QFT imply that $\mathfrak{M}(\mathcal{O})$ is a Type-III factor (Araki 1964); and results by Buchholz *et al.* (1987) indicate that the Type-III character of local algebras holds not only for free scalar fields but quite generically for quantum fields of physical interest.[12] By contrast, the global von Neumann algebra will be Type I if the representation satisfies the Spectrum Condition (the energy–momentum operator has a spectrum confined to the future light cone) and contains a vacuum state (a cyclic vector invariant under space-time translations), or even if there is no vacuum state but there is a 'mass gap.' However, there are massless theories of both bosons and fermions with no mass gap and no vacuum state in which the global von Neumann algebra satisfies a positive-energy condition but is Type II or Type III (see Doplicher *et al.* 1984, Buchholz & Doplicher 1984, and Borek 1985).

[11] Where $\mathfrak{M}$ is a collection of operators acting on the Hilbert space $\mathcal{H}$, its *commutant* $\mathfrak{M}'$ consists of every element of $\mathfrak{B}(\mathcal{H})$ that commutes with every element of $\mathfrak{M}$. See App. A for details.

[12] More particularly, the local algebras will be hyperfinite factors of Type III_1. But such niceties will play no role here.

Non-Type-I algebras are also commonplace for QSM in the thermodynamic limit, where the number of constituents of the system and its volume tend to infinity while the density remains finite. In this limit the so-called Kubo–Martin–Schwinger (KMS) condition explicates the notion of equilibrium (for a brief exposition, see Sewell 1986, pp. 49–51). KMS states at finite temperatures in the thermodynamic limit of QSM correspond to Type-III factors for a wide variety of physically interesting systems: Bose and Fermi gases, the Einstein crystal, the BCS model (see Emch 1972, pp. 139–40; Bratteli & Robinson 1997, Cor. 5.3.36). The exceptions are KMS states at temperatures at which phase transitions occur (if there are any for the systems in question); then the relevant algebras are direct sums/integrals of Type-III factors. Systems in equilibrium at *infinite* temperatures are also of interest in QSM. Such systems occupy chaotic states. Chaotic states in the thermodynamic limit of QSM correspond to Type-II_1 factors (see Takesaki 2003, Vol. 3, Sec. XIV.1).

We will be concerned here mainly with Type-III algebras, but many of the remarks below apply also to Type-II algebras.[13]

3.2 *Novel features of Type-III algebras*

We list here some features of Type-III algebras that may be unfamiliar to philosophers who work on foundations of ordinary QM and, thus, are used to the properties of Type-I algebras. As App. B elaborates, a defining feature of Type-III factor von Neumann algebras is that all their nontrivial projection operators are *infinite*. Roughly, the projection $E \in \mathfrak{M}$ is infinite if there is some nonzero projection F in $\mathfrak{M}$ such that F's range is a proper subspace of E's range and isometrically embeddable into E's range. It follows that Type-III factor algebras lack atoms. In general,

(i) Type-III algebras contain no atoms or minimal projectors.

(ii) Type-III algebras do not admit normal pure states. Any von Neumann algebra—or, for that matter, any C^*-algebra—admits pure states. In fact, if A is a positive element of a C^*-algebra $\mathcal{A}$ then there is a pure state ω on $\mathcal{A}$ such that $\omega(A) = \|A\|$. But for Type-II and -III von Neumann algebras there are no *normal* pure states. The proof is by *reductio*: the support projection for a normal pure state would have to be an atom, but by (i) there are no atomic projectors in a Type-III algebra.

From (ii) and Fact 1 it follows that

(iii) normal states on Type-III algebras have no filters.

[13] For an accessible discussion of the physical implications of Type-III algebras in QFT, see Yngvason 2005. Section 5.2.2 will discuss $\mathfrak{R}_Q$, a non-Type-I algebra related to the position-observable of ordinary QM. Halvorson 2001 gives an illuminating treatment of $\mathfrak{R}_Q$.

(iv) In the form familiar for $\mathfrak{B}(\mathcal{H})$, the superposition principle holds that if one forms a linear combination of pure vector states, one obtains a pure vector state. The superposition principle is subverted for Type-III algebras. Vector states on $\mathfrak{B}(\mathcal{H})$ are pure states, and the linear combination of two such states results in another pure vector state. The superposition principle is compromised in ordinary QM by the presence of superselection rules. In such cases the algebra of observables has the form $\bigoplus_j \mathfrak{B}(\mathcal{H}_j)$, and the linear combination of some pure vector states—those that belong to different superselection sectors—results in a vector state that is mixed rather than pure. But, of course, within a superselection sector superposition works per usual. For Type-III algebras, however, no vector state is pure (since vector states are normal and by (ii) normal states are impure). Thus, for Type-III algebras the antecedent of the superposition principle—'If one forms a linear combination of two pure vector states ...'—is never fulfilled.

(v) Eigenvalues of self-adjoint elements of a Type-III algebra are infinitely degenerate.

Features (i)–(iv) also obtain for Type-II algebras. Feature (i) and its consequences hold for some Type-I non-factor algebras as well;[14] a concrete example will be given in Sec. 5. But in physics typical examples of atomless algebras concern Type-II or Type-III algebras. Atomless Type-I non-factor algebras arise in making interpretational moves, such as understanding the nature of continuous quantum magnitudes (see Halvorson 2004) or in implementing the maximal-abelian strategy discussed below for interpreting quantum probabilities.

The features listed above unmoor a number of fixed points on the conventional (that is, Type-I factor) quantum interpretation horizon. The interpreter accustomed to that horizon may be forgiven for experiencing vertigo. The next section offers a remedy in the form of fixed points on that horizon that remain in the general setting.

4 When $\mathfrak{M}$ is not Type I: Probability formalism

Does an analogue of Gleason's Theorem hold for von Neumann algebras not isomorphic to $\mathfrak{B}(\mathcal{H})$? The answer is positive, but nearly twenty years of work were needed to work out all the details.

Generalized Gleason's Theorem (Maeda 1989; Hamhalter 2003, Ch. 5) *Let $\mathfrak{M}$ be a von Neumann algebra acting on a (separable) Hilbert space $\mathcal{H}$ and let $\mu\colon \mathcal{P}(\mathfrak{M}) \to [0,1]$ be a countably additive probability measure. If $\mathfrak{M}$ does not contain any summands*

[14] The Type-I non-factor algebras involved in superselection rules do possess atoms.

of Type I_2, there is a unique normal state ω on $\mathfrak{M}$ such that $\omega(E) = \mu(E)$ for all $E \in \mathcal{P}(\mathfrak{M})$.[15]

In the case where $\mu\colon \mathcal{P}(\mathfrak{M}) \to [0,1]$ is only finitely additive, the result is that there is a unique singular state ω on $\mathfrak{M}$ such that $\omega(E) = \mu(E)$ for all $E \in \mathcal{P}(\mathfrak{M})$.

Next we note that the motivation for Lüders' Rule, originally formulated for the case of ordinary quantum mechanics with $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ (cf. Sec. 2), carries over to all von Neumann algebras for which the generalized Gleason Theorem applies. That is, the Lüders-conditionalized state ω_E is the unique normal state satisfying the property (L) of Sec. 2.1.[16] To the extent that this motivation is persuasive for ordinary QM and Type-I algebras, it is also persuasive for other von Neumann algebras.

The discussion of Sec. 2 confined its attention to Lüders' Rule for a *selective* measurement. Here we will note a difficulty with applying Lüders' Rule for *nonselective* measurements to observable-algebras that are not Type I_n for n finite. Consider a nonselective measurement of O. In the case (a) that O has a pure discrete spectrum, let $\{E_i\}$ be O's spectral projectors. Then, given a pre-measurement state ω on a von Neumann algebra $\mathfrak{M}$ and a nonselective measurement of O, the obvious way to define the post-measurement state is by a natural generalization of Lüders' Rule:

$$\omega_O(A) \ := \ \frac{\omega(\sum_i E_i A E_i)}{\omega(\sum_i E_i)} \ = \ \sum_i \omega(E_i A E_i).$$

So far, so good. But now consider the case (b), that the spectrum is (partly) continuous. One could try to replace the sum by an integral, but there is no reason to think that the integral will always converge (see Davies 1976, Sec. 4.4).

Here we may have recourse to the literature on the existence of conditional expectations on a von Neumann subalgebra $\mathfrak{N} \subset \mathfrak{M}$. Go back to the case (a) where $O \in \mathfrak{M}$ has a pure discrete spectrum. Consider the Boolean subalgebra $\mathfrak{N} := \{\, B \in \mathfrak{M}\colon BE_i = E_i B \text{ for all } i \,\}$. Let T^O be the map from $\mathfrak{M}$ to $\mathfrak{N}$ given by $T^O(A) = \sum_i E_i A E_i$. Then our nonselective Lüders conditionalization rule can be written $\omega_O(A) = \omega\big(T^O(A)\big)$. In general, for arbitrary $\mathfrak{N} \subset \mathfrak{M}$, we will define the *conditional expectation of $\mathfrak{M}$ into $\mathfrak{N}$ determined by ω* with a linear map $T\colon \mathfrak{M} \to \mathfrak{N}$ such that (i) $T(A) \geq 0$ for positive $A \in \mathfrak{M}$, (ii) $T(I) = I$, (iii) $\omega\big(T(A)\big) = \omega(A)$ for all $A \in \mathfrak{N}$, and (iv) $T\big(AT(B)\big) = T(A)\,T(B)$ for all A, $B \in \mathfrak{N}$. Conditional

[15] A von Neumann algebra is of Type I_n if the unit element can be written as the sum of n abelian projectors. When $\mathcal{H}$ is nonseparable the result continues to hold if countable additivity is replaced by complete additivity.

[16] **Proof** Let φ be a normal state on a von Neumann algebra $\mathfrak{M}$ with said property. Consider an arbitrary $G \in \mathcal{P}(\mathfrak{M})$. $G = GE + GE^{\perp}$. $\omega_E(G) = \omega_E(GE) + \omega_E(GE^{\perp})$, and similarly for φ. Since $GE \leq E$, $\omega_E(GE) = \varphi(GE)$. And $\omega_E(GE^{\perp}) = \varphi(GE^{\perp}) = 0$ (because $\varphi(GE^{\perp}) \leq \varphi(E^{\perp}) = 0$). Thus, $\omega_E(G) = \varphi(G)$. But since a normal state is determined by its action on $\mathcal{P}(\mathfrak{M})$, $\omega_E = \varphi$. □

expectations can be well defined even when Lüders' Rule is not. Accardi & Cecchini (1982) prove that such a map always exists when ω is a faithful normal state (cf. p. 287 for faithfulness). It follows from a theorem of Takesaki (1972), again assuming that ω is a faithful normal state, that the map T is given by a norm-1 projection of $\mathfrak{M}$ onto $\mathfrak{N}$ iff the modular automorphism group σ_t^{ω} determined by ω preserves $\mathfrak{N}$, i.e. $\sigma_t^{\omega}(\mathfrak{N}) = \mathfrak{N}$ for all $t \in \mathbb{R}$.[17] For applications to Bayesian statistical inference, see Rédei 1992 and Valente 2007.

In sum, the fact that in QFT and QSM one has to deal with non-Type-I von Neumann algebras does not mean that the formalism of quantum probabilities developed in ordinary QM for Type-I algebras has to be abandoned or significantly modified.

5 When $\mathfrak{M}$ is not Type I: Probability interpretation

5.1 *Preparation*

Part of the task of interpreting quantum probability is making sense of our capacity to bring quantum systems into conditions we can understand as assigning probabilities. This task is entangled with the problem of quantum state preparation. In the Type-I case, interpretations typically account for our capacity to prepare a quantum state φ by appeal to Lüders' Rule (or the Projection Postulate) and the presence in $\mathfrak{B}(\mathcal{H})$ of a filter E_φ for φ. Although Lüders' Rule extends to the Type-III case, the availability of filters does not. Fact 1 along with Novel Feature (ii) are to blame: the only normal states on a von Neumann algebra admitting filters are pure ones, and a Type-III algebra $\mathfrak{M}$ has no normal pure states. This renders the account of preparation by filtration bankrupt.

Local relativistic QFT may have a way to compensate for the lack of filters for normal states of Type-III algebras. As noted in Sec. 3, the local algebra $\mathfrak{M}(\mathcal{O})$ associated with an open bounded region $\mathcal{O}$ of space-time is generically Type III. Therefore, any normal state on $\mathfrak{M}(\mathcal{O})$ is a mixed state which does not have a filter in $\mathfrak{M}(\mathcal{O})$. Thus, there can be no local preparation procedure for a normal state on $\mathfrak{M}(\mathcal{O})$ that consists in measuring a filter in $\mathfrak{M}(\mathcal{O})$. Fortunately, however, the standard axioms for local relativistic QFT imply that the funnel property holds for suitable space-time regions in certain models.[18] The funnel property entails that any normal state on $\mathfrak{M}(\mathcal{O})$ does have a filter in some $\mathfrak{M}(\widehat{\mathcal{O}})$ where $\widehat{\mathcal{O}} \supset \mathcal{O}$, guaranteeing that some local preparation procedure is possible, albeit in an expanded sense of 'local' (see Buchholz *et al.* 1987).

Whether it is desirable to thus expand our sense of 'local' is perhaps a matter for debate. But it is clear that the QFT stratagem for securing preparation by

[17]Modular automorphism groups will be discussed below, in Sec. 5.2.4.

[18]The net of local algebras $\mathfrak{M}(\mathcal{O})$ have the *funnel property* if and only if for any open bounded $\mathcal{O}$ there is another open bounded $\widehat{\mathcal{O}} \supset \mathcal{O}$ and a Type-I factor $\mathfrak{N}$ such that $\mathfrak{M}(\widehat{\mathcal{O}}) \supset \mathfrak{N} \supset \mathfrak{M}(\mathcal{O})$.

filtration cannot be adapted to the setting of the thermodynamic limit of QSM. What one would like to be able to prepare is a state of a superconductor or a ferromagnet, that is, a state of the *entire* quasi-local algebra itself. That algebra will typically be Type III, and no funnel property can be invoked to embed it in a Type-I algebra. We see no prospect of a filtration-based account of the preparation of normal states in the case of QSM. Nor does it appear promising to aim instead at an account of the preparation of singular (non-normal) states. For one thing, the aim of preparation is to bring a system into a state from which we can extract probabilistic predictions that enable us to make sense of the natural world. Insofar as the probabilities assigned by singular states lack coherence of countable additivity and may also, Sec. 2.1 suggested, lack the coherence of instantiating natural laws, accounting for the preparation of singular states is a Pyrrhic victory. For another thing, the account of preparation by filtration may not extend intact to singular states. The very notion of a filter supposes that the state of a system prior to a filtration-interaction is normal. Once singular states are countenanced, this supposition is unmotivated.

In sum, there are reasons to doubt that strategies for making sense of state preparation in the Type-I case will succeed in a general setting. But let us bracket these doubts for now. Let us turn to the question of how to understand the probabilities assigned by a state ω on a Type-III von Neumann algebra $\mathfrak{M}$.

5.2 *Interpretation of probabilities*

We will work with a template for the interpretation of quantum probability which, we contend, continues to apply even to atomless von Neumann algebras. We call the template the Maximal-Beable Approach (MBA). The nomenclature is due to John Bell, who envisioned a future theory that is more satisfactory than the present quantum theory in that it does not appeal to the unanalysed concept of measurement:

> Such a theory would not be fundamentally about 'measurement', for that would again imply incompleteness of the system and unanalyzed interventions from outside. Rather it should again become possible to say of that system not that such and such may be *observed* to be so but that such and such *be* so. The theory would not be about *observ*ables but about '*be*ables'. (Bell 1987c, p. 41)

Attempts have been made to understand the present quantum theory in terms of beables. There are two ways to unpack this notion, both of which can be construed as ways to get at the idea that, relative to a state φ on a von Neumann algebra of observables $\mathfrak{M}$, a subalgebra $\mathfrak{R} \subseteq \mathfrak{M}$ consists of beables if its elements can be assigned simultaneously definite values, the probabilities of which are defined by φ. Both approaches lead to abelian subalgebras of the von Neumann algebra of interest.

5.2.1 *Classical probability models* The first approach to identifying beable subalgebras starts with C^*-algebras before specializing to von Neumann algebras. The idea is that the C^*-algebra $\mathcal{A}$ counts as a beable algebra for the state φ if the pair $\varphi, \mathcal{A}$ admits a *classical probability model*. For an abelian algebra such a model consists of a probability space (Λ, μ_φ) with $\int_\Lambda d\mu_\varphi = 1$ and an association $\mathcal{A} \ni Z \longmapsto \hat{Z}$, where $\hat{Z}$ is a measurable complex-valued function on Λ, such that

(C1) $\varphi(A) = \int_\Lambda \hat{A}(\lambda)\, d\mu_\varphi(\lambda)$ for any $A \in \mathcal{A}$

and

(C2) $\widehat{AB}(\lambda) = \hat{A}(\lambda)\,\hat{B}(\lambda)$, $\big(\widehat{A+B}\big)(\lambda) = \hat{A}(\lambda) + \hat{B}(\lambda)$ and $\widehat{A^*}(\lambda) = \hat{A}^*(\lambda)$ for all $\lambda \in \Lambda$ and all $A, B \in \mathcal{A}$.

Condition (C1) says that the elements of $\mathcal{A}$ can be interpreted as random variables on a common probability space and the expectation values assigned by φ can be interpreted as weighted averages of the random variables. Condition (C2) requires that the association $Z \longmapsto \hat{Z}$ preserves the structure of the algebra. (C2) has been labeled a noncontextuality assumption as well as a causality assumption. The latter designation seems to us dubious, but we will not press the point here. Any abelian C^*-algebra containing the identity, and, *a fortiori*, any abelian von Neumann algebra, admits a classical probability model for any state.

Theorem 1. (Gelfand) *Let $\mathcal{A}$ be a unital abelian C^*-algebra. $\mathcal{A}$ is isomorphic to the algebra $C_0(X)$ of the continuous complex-valued functions on a compact Hausdorff space X.*

The proof of this theorem (see Bratteli & Robinson 1987, Thm 2.1.11A) provides the basis for a classical probability setup. The sample space Λ consists of the pure states on $\mathcal{A}$ equipped with the weak* topology inherited from the dual $\mathcal{A}^*$ (see App. A). The elements of $\mathcal{A}$ become random variables on Λ via the Gelfand transform $A \longmapsto \hat{A}$, where $\hat{A}(\omega) := \omega(A)$ for $\omega \in \Lambda$. A pure state ω on an abelian $\mathcal{A}$ is multiplicative (Kadison & Ringrose 1997a, Prop. 4.4.1), i.e. $\omega(AB) = \omega(A)\,\omega(B)$ for all $A, B \in \mathcal{A}$, whence $\widehat{AB}(\omega) = \omega(AB) = \omega(A)\,\omega(B) = \hat{A}(\omega)\,\hat{B}(\omega)$. And since $\big(\widehat{A+B}\big)(\omega) = \omega(A+B) = \omega(A) + \omega(B) = \hat{A}(\omega) + \hat{B}(\omega)$ and $\widehat{A^*}(\omega) = \omega(A^*) = \omega(A)^* = \hat{A}^*(\omega)$, condition (C2) is satisfied. Furthermore, the Riesz Representation Theorem shows that states on $C_0(\lambda)$ are in one–one correspondence with Borel probability measures on Λ. Thus, for any state φ on a unital abelian C^*-algebra $\mathcal{A}$ there is a Borel probability measure μ_φ on Λ such that

$$\varphi(ABC\cdots) = \int_\Lambda \hat{A}(\omega)\,\hat{B}(\omega)\,\hat{C}(\omega)\cdots\, d\mu_\varphi(\omega) \qquad \text{for all } A, B, C, \ldots \in \mathcal{A}.$$

When $\mathcal{A}$ is non-abelian it seems reasonable to continue to require (C2) for commuting elements of $\mathcal{A}$. With this understanding it can be shown that on the present approach to beables, a non-abelian $\mathcal{A}$ does not count as a beable algebra relative to some states φ if $\mathcal{A}$ is a von Neumann algebra that is rich enough to describe Bell–EPR-type experiments by containing EPR–Bell operators. Such operators have the form $1/2\left[X_1(Y_1+Y_2)+X_2(Y_1-Y_2)\right]$ where the X_i and Y_j $(i, j = 1, 2)$ are self-adjoint and $-I \leq X_i, Y_j \leq I$ and $[X_i, Y_j] = 0$ but $[X_1, X_2] \neq 0$ and $[Y_1, Y_2] \neq 0$. For a state φ on such an algebra define $\beta(\varphi) := \sup_B\left|\varphi(B)\right|$. A state $\varphi_{\not\leq}$ is said to violate the Bell inequalities if $\beta(\varphi_{\not\leq}) > 1$. The results of Fine (1982b) show that if $\mathfrak{M}_{\text{Bell}}$ is a von Neumann algebra rich enough to describe Bell–EPR experiments and $\varphi_{\not\leq}$ is a Bell-inequality-violating state, then $\varphi_{\not\leq}, \mathfrak{M}_{\text{Bell}}$ does not admit a classical probability interpretation. Thus, on the present approach to explicating beables, $\mathfrak{M}_{\text{Bell}}$ does not count as a beable algebra relative to $\varphi_{\not\leq}$. In relativistic QFT, states that violate the Bell inequalities are endemic (see Halvorson & Clifton 2000).

5.2.2 *Mixtures of dispersion-free states* A second approach to beables (advocated by Halvorson & Clifton 1999) is to count a C^*-algebra $\mathcal{A}$ as a beable algebra relative to the state φ just in case φ can be represented as a mixture of dispersion-free states on $\mathcal{A}$, i.e. there is a measure $\mu_\varphi(\omega)$ on the set D of dispersion-free states (i.e. states ω such that $\omega(A^2) = \left(\omega(A)\right)^2$ for all $A \in \mathcal{A}$) such that

MDFS $\varphi(A) = \int_D \omega(A)\, \mu_\varphi(\omega)$ *for all* $A \in \mathcal{A}$.

Since a pure state on an abelian $\mathcal{A}$ is multiplicative and, thus, dispersion-free it follows from the previous section that any state on an abelian algebra can be represented as a mixture of dispersion-free states. Halvorson & Clifton (1999) prove a partial converse by showing that if a faithful state φ on a C^*-algebra $\mathcal{A}$ can be represented as a mixture of dispersion-free states then $\mathcal{A}$ must be abelian. For our purposes, it is reasonable to restrict attention to faithful states because most physically significant states in QFT and QSM are faithful. Interpretations that falter with respect to faithful states are thus unacceptable, and those that succeed make a strong case for acceptance.

5.2.3 *The* MBA The results of the two preceding subsections encourage the idea that (a) relative to any state φ an abelian C^*-algebra $\mathcal{A}$ counts as a beable algebra, and (b) if φ is a faithful state, or else $\mathcal{A}$ is a von Neumann algebra rich enough to describe Bell–EPR experiments and φ is a Bell-inequality-violating state, then in order for $\mathcal{A}$ to count as a beable algebra relative to φ it must be abelian. For here on we will ignore the qualifications in (b) and simply assume that for a von Neumann algebra to count as a beable algebra it must be abelian.

Having decided that a beable subalgebra $\mathfrak{R}$ of a von Neumann algebra $\mathfrak{M}$ is an abelian subalgebra, it is natural to look for a maximal abelian subalgebra. In implementing this strategy it is important to parse the relevant maximality as follows: $\mathfrak{R}$ is abelian (i.e. $\mathfrak{R} \subseteq \mathfrak{R}'$; recall that '$'$' denotes 'commutant'), and maximally so with respect to $\mathfrak{M}$ (i.e. $\mathfrak{R}' \cap \mathfrak{M} \subseteq \mathfrak{R}$) with the upshot that $\mathfrak{R} =$

$\mathfrak{R}' \cap \mathfrak{M}$. When $\mathfrak{M} \neq \mathfrak{B}(\mathcal{H})$ (where $\mathcal{H}$ is the Hilbert space on which $\mathfrak{M}$ acts) it is unreasonable to require that $\mathfrak{R}$ is maximal with respect to $\mathfrak{B}(\mathcal{H})$, i.e. $\mathfrak{R}' \subseteq \mathfrak{R}$ and, consequently, $\mathfrak{R} = \mathfrak{R}'$. In any case such a beast may not exist, as follows from

Lemma 1 *A von Neumann algebra $\mathfrak{M}$ acting on $\mathcal{H}$ contains an abelian subalgebra maximal with respect to $\mathfrak{B}(\mathcal{H})$ iff $\mathfrak{M}'$ is abelian.*[19]

But if $\mathfrak{M}'$ is abelian then $\mathfrak{M}$ is Type I.[20] On the other hand, any von Neumann algebra $\mathfrak{M}$ contains abelian subalgebras that are maximal in $\mathfrak{M}$.[21] So with this understanding of maximality, the MBA to be described below can be applied to any von Neumann algebra.

Focusing on a maximal abelian subalgebra $\mathfrak{R} \subset \mathfrak{M}$ and its dispersion-free states has an attractive *semantic* consequence. The projection lattice $\mathcal{P}(\mathfrak{R})$ of an abelian $\mathfrak{R}$ is a Boolean lattice, admitting two-valued homomorphisms, maps from $\mathcal{P}(\mathfrak{R})$ to $\{0, 1\}$ preserving the classical truth tables (see App. C and Bell & Machover 1977 for more on lattice theory). A dispersion-free state on $\mathfrak{R}$ defines such a two-valued homomorphism, and conversely. Every self-adjoint element of $\mathfrak{R}$ has a spectral resolution in $\mathcal{P}(\mathfrak{R})$; thus a two-valued homomorphism on $\mathcal{P}(\mathfrak{R})$ induces a map taking each self-adjoint element of $\mathfrak{R}$ to one of its eigenvalues—a 'maximal set of observables with simultaneously determinate values' (Halvorson & Clifton 1999, p. 2442). It corresponds as well to a collection of determinate-value-attributing propositions that can receive truth valuations obedient to the classical truth tables.

We now have the ingredients for a scheme for casting not only quantum probabilities but also quantum semantics in a classical mold. We express the scheme succinctly:

Maximal-Beable Recipe *Given a system whose algebra of observables is $\mathfrak{M}$ and whose state is φ,*

Step 1: Identify a maximal abelian subalgebra $\mathfrak{R}$ of $\mathfrak{M}$.

Step 2: Characterize two-valued homomorphisms of $\mathcal{P}(\mathfrak{R})$. Each corresponds to a possible configuration of the system.

Step 3: Use φ to define a probability distribution over the homomorphisms identified in 2.

Step 2 of the recipe addresses what Sec. 2 called the problem of characterization while Step 3 addresses the problem of explication. We will discuss how these steps are carried out for Type-I and Type-III algebras.

[19]See Jauch 1960 and Jauch & Misra 1961.

[20]This is because $\mathfrak{M}$ and $\mathfrak{M}'$ are of the same type. Bear in mind that a Type-I algebra is not necessarily a Type-I factor or even a direct sum of Type-I factors.

[21]**Proof** Note that the abelian subalgebras of any von Neumann algebra $\mathfrak{M}$ are partially ordered by inclusion. Then apply Zorn's Lemma. □

The MBA *in Action: Type-I case* There is a simple procedure for generating a maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$. One starts with a complete set $\{E_i\}$ of orthogonal one-dimensional projection operators—that is, atoms—in $\mathfrak{B}(\mathcal{H})$ and one closes in the weak topology. The result will be an abelian von Neumann algebra including every self-adjoint element of $\mathfrak{B}(\mathcal{H})$ that has $\{E_i\}$ as a spectral resolution (see Beltrametti & Cassinelli 1981, Sec. 3.2). Call this the maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$ generated by $\{E_i\}$. Familiar interpretations of ordinary QM can be seen as trafficking in maximal abelian subalgebras of $\mathfrak{B}(\mathcal{H})$ so obtained. In collapse interpretations, the $\{E_i\}$ are eigenprojections of the observable measured. In modal interpretations, the $\{E_i\}$ are eigenprojections of the density operator W giving the state of the system. In 'Bohmian' interpretations, they are eigenprojections of the preferred determinate observable. In each case, the algebra generated is maximal abelian if and only if each E_i is an atom.

Each atom in a maximal abelian subalgebra $\mathfrak{N}$ determines a two-valued homomorphism on its projection lattice $\mathcal{P}(\mathfrak{N})$,[22] and thus an assignment of eigenvalues to self-adjoint elements of $\mathfrak{N}$. Such an *eigenvaluation* is, in Bub's (1997, p. 18) words, 'a maximal set of co-obtaining properties.' The atom defining a homomorphism determines (via the trace prescription) the probability a state W on $\mathfrak{B}(\mathcal{H})$ assigns the eigenvaluation corresponding to that homomorphism. This atomic strategy solves the problems of characterization and explication while resurrecting a classical probability structure (the trace prescription restricted to $\mathcal{P}(\mathfrak{N})$) and a classical semantic structure ($\mathcal{P}(\mathfrak{N})$ understood as a lattice of propositions) from the ashes of QM's assault on our time-honored intuitions.

The MBA *in Action: Non-Type-I case* Familiar variations on the MBA for ordinary QM use atoms in the projection lattice of $\mathfrak{B}(\mathcal{H})$ to pick out a maximal beable subalgebra for a system in the state φ on $\mathfrak{B}(\mathcal{H})$. These same atoms explicate the probabilities φ assigns possible value states of the system, according to the MBA: where E is the atom coding a value state, $\varphi(E)$ gives the probability of that value state obtaining. A *prima facie* impediment to extending such interpretations to an arbitrary von Neumann algebra $\mathfrak{M}$ is the possible absence from $\mathfrak{M}$ of atoms; for if $\mathfrak{M}$ lacks atoms then so does any abelian subalgebra of $\mathfrak{M}$ that is maximal abelian with respect to $\mathfrak{M}$ (that is, properly contained in no abelian subalgebra of $\mathfrak{M}$).[23]

[22] Where E is the atom, the homomorphism is $h(F) = 1$ if $E < F$; $h(F) = 0$ otherwise. If $\mathcal{P}(\mathfrak{N})$ is finite, *all* its two-valued homomorphisms are determined in this way (Bell & Machover 1977, Cor. 5.3).

[23] *Outline of proof:* Suppose that $\mathfrak{M}$ is atomless, and that $\mathfrak{N}$ is a maximal abelian subalgebra with respect to $\mathfrak{M}$. Now suppose, for *reductio*, that E is an atom in $\mathcal{P}(\mathfrak{N})$. Because $\mathfrak{M}$ contains no minimal projections, there exists $F \in \mathfrak{M}$ such that $F < E$. Because E is an atom in $\mathcal{P}(\mathfrak{N})$, $F \notin \mathcal{P}(\mathfrak{N})$, and hence $F \notin \mathfrak{N}$. But then $\mathfrak{N}$ is not a *maximal* abelian subalgebra with respect to $\mathfrak{M}$, which is our contradiction. To see that $\mathfrak{N}$ is not a *maximal* abelian subalgebra with respect to $\mathfrak{M}$, consider the algebra $\mathfrak{N} \cup F$. Because $F < E$ and E commutes with every element of $\mathcal{P}(\mathfrak{N})$, F commutes with every element of $\mathcal{P}(\mathfrak{N})$. Because $\mathcal{P}(\mathfrak{N})$ generates $\mathfrak{N}$, it follows that F commutes with every element of $\mathfrak{N}$. So $\mathfrak{N} \cup F$,

But the impediment is only *prima facie*. Nothing in the maximal-beable recipe, above, requires us to specify maximal abelian subalgebras, code facts, or mediate probability assignments, by appeal to atoms. And in the Type-III case, we will have all the ingredients the recipe calls for. As already noted, any von Neumann algebra $\mathfrak{M}$ contains abelian subalgebras that are maximal in $\mathfrak{M}$. This is all Step 1 of the recipe requires. Now if ω is dispersion-free on such a subalgebra $\mathfrak{R}$, then $\omega(A) \in \text{Sp}(A)$ for all self-adjoint $A \in \mathfrak{R}$. In particular, ω takes every element of the projection lattice $\mathcal{P}(\mathfrak{R})$ to the spectrum $\{0,1\}$ characteristic of a projection operator. $\mathcal{P}(\mathfrak{R})$ is a Boolean lattice on which ω thereby defines a two-valued homomorphism. This gets us, without the mediation of atoms, the existence of two-valued homomorphisms Step 2 of the recipe calls for. We also saw that any state φ on an abelian subalgebra $\mathfrak{R}$ of an arbitrary $\mathfrak{M}$ can be expressed as a mixture of dispersion-free states; hence, φ corresponds to a probability distribution over these homomorphisms, which is all Step 3 of the recipe calls for. Despite the proclivities of variations developed for ordinary QM, the MBA does not presuppose that $\mathfrak{R}$ contains atoms.

However, while the nonatomic pursuit of the MBA is formally possible, it has its drawbacks. For one thing, mixtures of dispersion-free states on maximal abelian subalgebras of a Type-III (or -II) algebra are mixtures of non-normal states. This follows because (a) these subalgebras are atomless and, therefore, do not admit normal pure states and (b) for any von Neumann algebra, normal + dispersion-free implies pure.[24] Insofar as normal states are the physically realizable ones, this result threatens to undermine the interpretative strategy.

Perhaps the MBA can navigate this bump by observing that normality is a virtue states exhibit in exercising their capacity to assign probabilities; it is the virtue of assigning countably additive probabilities. But from the point of view of MBA the singular dispersion-free states ω appearing in the (MDFS) decomposition of a state φ (see Sec. 5.2.2) on an abelian $\mathfrak{R} \subset \mathfrak{M}$ are not assigning probabilities; rather, they are defining two-valued homomorphisms on $\mathfrak{R}$. The singular states ω are also *receiving* (on behalf of those homomorphisms and the maximal patterns of beable instantiation they determine) probabilities from φ. Their non-normality need not hinder them in either of these roles.

Still, it prompts a question. What are the maximal patterns of beable instantiation coded by these singular dispersion-free states ω *like*? This is a fair question, for its answer is a solution to the problem of characterization. It is also a vexed

which strictly contains $\mathfrak{R}$, is abelian.

[24]**Proof** Let $\mathfrak{M}$ be a von Neumann algebra and suppose ω is dispersion-free on its projection lattice $\mathcal{P}(\mathfrak{M})$: for each projection $E \in \mathfrak{M}$, either $\omega(E) = 1$ or $\omega(E) = 0$. But then ω must be pure on $\mathcal{P}(\mathfrak{M})$: if E is a projection operator and $\{0,1\} \ni \omega(E) = \lambda\omega_1(E) + (1-\lambda)\,\omega_2(E)$ for $0 \leq \lambda \leq 1$, then either $\lambda \in \{0,1\}$ or $\omega_1 = \omega_2$. Either way, ω is pure. Because ω is normal, its extension to $\mathfrak{M}$ is determined by its action on $\mathcal{P}(\mathfrak{M})$. Purity on the latter implies purity on the former. □

question. To see why, we will consider an example of a dispersion-free state on an atomless maximal abelian von Neumann algebra. Let $\mathcal{H}$ be the separable Hilbert space $L_2(0,1)$ of square integrable functions on the unit interval $(0,1)$ equipped with the Lebesgue measure. Where f is a bounded measurable function on $(0,1)$, let M_f be the operator on L_2 corresponding to multiplication by f. The collection of such operators (with addition and multiplication defined pointwise) is a von Neumann algebra that is a maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$ (Kadison & Ringrose 1997b, Ex. 5.1.6 and p. 557). We will label it $\mathfrak{R}_Q$ because its projection operators are characteristic functions χ_Δ for Borel subsets Δ of $(0,1)$, and we can think of such functions as eigenprojections of a position observable Q for a point particle confined to the unit interval. $\mathfrak{R}_Q$ has no minimal projection operators. (*Intuitive argument:* The only nonzero projection operators in $\mathfrak{R}_Q$ are characteristic functions χ_Δ for sets Δ not of measure 0. $\chi_\Delta < \chi_{\Delta'}$ if $\Delta \subset \Delta'$. Because any measurable set has a measurable proper subset, no projection in $\mathfrak{R}_Q$ is minimal; see Kadison & Ringrose 1997b, Lem. 8.6.8.)[25]

To investigate the character of dispersion-free states on $\mathfrak{R}_Q$, we will exploit (without explicating) two important facts.

Fact 2 (Ultrafilters generate two-valued homomorphisms) *Each ultrafilter of a Boolean lattice generates a two-valued homomorphism on that lattice.*

Fact 3 (Ultrafilter Extension Theorem) *Any subset S of a Boolean lattice possessing the* finite-meet property*—viz. that if $x_1, \ldots, x_n \in S$ then $x_1 \cap \ldots \cap x_n \neq 0$—is contained in some ultrafilter.* (Bell & Machover 1977, Cor. 3.8)

The proof of the Ultrafilter Extension Theorem invokes Zorn's Lemma, which is equivalent to the Axiom of Choice.

Now consider a countable set $F_p \subset \mathfrak{R}_Q$ of characteristic functions for (ever-shrinking) open neighborhoods around some $p \in (0,1)$. F_p possesses the finite-meet property: for any finite family $\{\chi_1, \chi_2, \ldots\} \subset F_\lambda$, $\bigwedge_{i=1}^n \chi_i \neq 0$. (The intersection is a lock to contain p, at least!) By defining a two-valued homomorphism on $\mathcal{P}(\mathfrak{R}_Q)$, an ultrafilter defines a dispersion-free state on $\mathfrak{R}_Q$. It follows from the Ultrafilter Extension Theorem that there is a dispersion-free state ω_p on $\mathfrak{R}_Q$ such that $\omega_p(\chi) = 1$ for all $\chi \in F_p$.

Notice that ω_p assigns 'located at p' probability 1. This seems as if it ought to be maximally specific information about the possible condition ω_p codes. But it isn't. Even by the lights of the MBA, there are truths to be had that we don't know enough about ω_p to get. For there are countably many distinct pure (thus dispersion-free) states on $\mathfrak{R}_Q$ that assign 'located at p' probability 1 (Halvorson 2001, Prop. 2). Thus, for an arbitrary $\chi \in \mathfrak{R}_Q$, we don't know whether $\omega_p(\chi) = 1$ or $\omega_p(\chi) = 0$.

[25] $\mathfrak{R}_Q$ provides an example of the kind promised above; namely, a Type-I non-factor algebra that lacks atoms.

Contrast the pursuit of the MBA in the context of ordinary QM. Where $\mathfrak{N}$ is a maximal abelian subalgebra of $\mathfrak{B}(\mathcal{H})$, and E is an atom in $\mathfrak{N}$, the state ω^E defined by $\omega^E(A) := \text{Tr}(AE)$ for all $A \in \mathfrak{N}$ is a dispersion-free state defining a two-valued homomorphism on the projection lattice $\mathcal{P}(\mathfrak{N})$. For each $F \in \mathcal{P}(\mathfrak{N})$, we know whether $\omega^E(F)$ is 0 or 1. Knowing the truth value of each proposition we take to be determinately truth-valued, we know exactly what a world coded by E is like.

It is also illuminating to note the contrast with a dispersion-free state (q, p) of a classical system with phase space $\mathbb{R}^2$. Functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ make up the beable algebra. For every element of this algebra, we can say what value the pure state (q, p) assigns it. Once again, the MBA sets terms for characterizing a possible world, which terms it is able to meet. In the presence of a nonatomic von Neumann algebra, this is not so. The problem is that we lack a *constructive* procedure for specifying a dispersion-free state on a nonatomic beable algebra. As Halvorson (2001, p. 41) aptly puts it, '[a]lthough we "know" that there are ultrafilters (i.e., pure states) [on such an algebra], we do not know this because someone has constructed an example.' We know it from a Zorn's Lemma argument.

The 'ineffability' of dispersion-free states on atomless $\mathcal{P}(\mathfrak{N})$ suggests that we have no handle, analogous to the one supplied in the Type-I case by applying the eigenstate–eigenvalue link to an atom defining a dispersion-free state, on how to decode the facts these dispersion-free states encode. It suggests that we have no handle, analogous to the one supplied in the Type-I case by applying the trace prescription to the system state and a coding atom, on how to assign those facts probabilities. Confronted with the atomless von Neumann algebras of QFT and the thermodynamic limit of QSM, then, the MBA shirks two key interpretive tasks. First, it fails to explicitly characterize the recipients of quantum probability assignments. Judged by its own lights, it doesn't complete the task of lending content to the possible conditions of a quantum system. Second, it fails to explicate the probabilities the theory assigns. Thus the MBA fails to equip QFT and the thermodynamic limit of QSM with empirical content, in the form of specific probability assignments to explicitly characterized conditions possible for systems described by atomless von Neumann algebras.[26]

Even if we can make our peace with this circumstance, a question remains: Without the help of atoms is there a motivated way to pick out a maximal abelian subalgebra? The next response to atomlessness we consider finds motivation by abandoning maximality.

[26]But see Halvorson 2001 for a discussion of the nonconstructibility of pure states on atomless algebras, along with ways to avoid the difficulties that nonconstructibility creates.

5.2.4 *Giving up on maximality* The idea here is not to be so greedy as to demand maximality; settle for finding a motivated way to pick out an abelian subalgebra $\mathfrak{R} \subset \mathfrak{M}$ without demanding that $\mathfrak{R}$ be maximal abelian in $\mathfrak{M}$. The modal interpretation as developed for ordinary QM is one implementation of this idea. As a first crude cut, the self-adjoint elements of the beable subalgebra of $\mathfrak{B}(\mathcal{H})$ for a normal state φ corresponding to the density operator W are identified as those that share an eigenbasis with W or, equivalently, commute with W. Transferring this idea to a general von Neumann algebra $\mathfrak{M}$, the beable subalgebra for the state φ is the centralizer $\mathfrak{C}_\varphi(\mathfrak{M})$ of φ as defined by $\mathfrak{C}_\varphi(\mathfrak{M}) := \{ A \in \mathfrak{M}\colon\ \varphi(AB) = \varphi(BA) \text{ for all } B \in \mathfrak{M} \}$.[27] However, there is no guarantee that the subalgebra so identified will satisfy the characteristic touted above as essential to beables—namely, abelianness. To overcome this problem, take the beable subalgebra to be $\mathcal{Z}(\mathfrak{C}_\varphi(\mathfrak{M})) := \mathfrak{C}_\varphi(\mathfrak{M}) \cap \mathfrak{C}_\varphi(\mathfrak{M})'$, the center of the centralizer of φ. (See Earman & Ruetsche 2005, pp. 567–8, for a sketch of why this reproduces the familiar modal interpretation in the Type-I case.) For the case when φ is a faithful normal state, Rob Clifton (2000) proved a beautiful result that homes in on $\mathcal{Z}(\mathfrak{C}_\varphi(\mathfrak{M}))$ as the abelian subalgebra of $\mathfrak{M}$ that is maximal with respect to the property of being identifiable using only φ and algebraic operations within $\mathcal{R}$. This prescription for picking out a (nonmaximal) abelian subalgebra leads to a dead end for many physically relevant states on Type-III algebras.

By the Tomita–Takesaki Theorem (see Kadison & Ringrose 1997b, Ch. 9.2), a faithful normal state φ on a von Neumann algebra $\mathfrak{M}$ has associated with it a unique automorphism group σ_t^φ $(-\infty \leq t \leq +\infty)$ with respect to which φ is a KMS state at inverse temperature -1. This theorem is fundamental for the development of QSM. However, the application we have in mind is not confined to QSM but applies quite generally. It proceeds via the following lemma: the centralizer $\mathfrak{C}_\varphi(\mathfrak{M})$ of a faithful normal state φ coincides with the invariants $\mathcal{I}_{\sigma_t^\varphi}(\mathfrak{M}) := \{ A \in \mathfrak{M}\colon \sigma_t^\varphi(A) = A \text{ for all } t \in \mathbb{R} \}$ of the modular group (Kadison & Ringrose 1997b, p. 617). For Type-III algebras and for a large class of faithful normal states on such algebras—e.g. ergodic states—$\mathcal{I}_{\sigma_t^\varphi}(\mathfrak{M}) = \mathbb{C}I$. It follows that $\mathcal{Z}(\mathfrak{C}_\varphi(\mathfrak{M})) = \mathbb{C}I$, i.e. the modal beables are trivial. Trying to escape this no-go result by attacking Clifton's Characterization Theorem is to no avail if one agrees that any generalization of the modal interpretation to QFT and QSM must preserve the condition that, whatever other requirements the modally determinate observables for a state φ must satisfy, they must belong to the centralizer of φ.

5.2.5 *Mining for atoms* A third response to interpretive problems attendant upon the atomlessness of von Neumann algebras endemic to QFT and QSM is to find in their vicinity atomic algebras yielding to established techniques of

[27] If W is the density operator corresponding to the normal state φ and if $A \in \mathfrak{M}$ commutes with W, then $\varphi(AB) = \mathrm{Tr}(ABW) = \mathrm{Tr}(BAW) = \varphi(BA)$ for any $B \in \mathfrak{M}$, i.e. $A \in \mathfrak{C}_\varphi(\mathfrak{M})$.

interpretation. This is the strategy of Dennis Dieks' (2000) modal interpretation of QFT. Dieks exploits the funnel property already mentioned (see Sec. 5.1 above) to characterize the physics of an open bounded region $\mathcal{O}$ not in terms of the Type-III algebra $\mathfrak{M}(\mathcal{O})$ pertaining to $\mathcal{O}$, but in terms of a Type-I algebra $\mathfrak{N}$ such that $\mathfrak{M}(\mathcal{O}) \subset \mathfrak{N} \subset \mathfrak{M}(\widehat{\mathcal{O}})$, where $\widehat{\mathcal{O}} \supset \mathcal{O}$.

Clifton (2000) has criticized this maneuver as arbitrary. The choice of $\widehat{\mathcal{O}}$ is arbitrary; having chosen it, arbitrary too is the choice of interpolating Type-I factors. We would add that the status of the interpolating Type-I factors is somewhat mysterious. In the standard approaches to local QFT Minkowski space-time, the local von Neumann algebra $\mathfrak{M}(\mathcal{O})$ associated with an open bounded region $\mathcal{O}$ with non-empty spacelike complement is a Type-III factor. The basic interpretational premise of these approaches is that elements of this algebra correspond to operations that can be performed locally in $\mathcal{O}$. Supposing $\widehat{\mathcal{O}}$ is finite, every open bounded region that interpolates between it and $\mathcal{O}$ will therefore be associated with a Type-III factor algebra. The interpolating Type-I factor $\mathfrak{N}$ is, therefore, the local algebra of no interpolating region. Although it can be mathematically associated with local space-time regions, the association lacks operational significance unless additional interpretational premises are introduced.

6 Conclusion

Interpretations of quantum probability have typically targeted the Type-I factor von Neumann algebras in whose terms the quantum theories most familiar to philosophers are formulated. But quantum physics makes use of von Neumann algebras of more exotic types, such as Type-III factors. Physics itself thus demands we extend our interpretation of quantum probability to these more exotic types. While significant features of the quantum probability formalism survive the extension, strategies for interpreting that formalism have not fared so well. State preparation is an exercise of our capacity to bring quantum systems into conditions that we can understand as *assigning* probabilities. Strategies adapted to Type-I von Neumann algebras for accounting for preparation exploit the presence in those algebras of filters. These strategies go extinct in the environment of Type-III factors, which lack filters. Strategies for prosecuting the MBA adapted to Type-I von Neumann algebras use atoms in the projection lattice of a von Neumann algebra $\mathfrak{M}$ to specify a maximal abelian subalgebra $\mathfrak{R}$ of $\mathfrak{M}$, to define homomorphisms on $\mathfrak{R}$'s projection lattice, and to explicate the quantum probabilities a state ω on $\mathfrak{M}$ assigns the eventualities coded by these homomorphisms. These strategies likewise go extinct in the environment of Type-III factor algebras, whose projection lattices are atomless. The MBA may be pursued nonatomically, because even an atomless $\mathfrak{M}$ has maximal abelian subalgebras whose projection lattices admit homomorphisms over which an arbitrary state on $\mathfrak{M}$ defines a probability measure. But the MBA adapted to the nonatomic case may not fare well in the struggle for interpretive existence. Without atoms, it is not clear how to make a principled selection of a maximal abelian subalgebra framing one's interpretation; even with such an algebra selected, its dispersion-free states are ineffable: by the MBA's own lights, there is more to say about a system whose condition is coded by such a state than we can at present constructively say. Clifton's modal interpretation of QFT gives up on maximality *tout court*. But the survival it thereby purchases is devoid of meaning for many physically significant states: applied to ergodic states, Clifton's QFT-adapted modal interpretation assigns determinate truth values only to tautologies and logical falsehoods. None of this is to say that in more exotic settings, quantum probabilities *defy* interpretation, even interpretation by adapting familiar strategies. We count what we have chronicled as resistance rather than defiance, resistance that invites further effort on the problem.

Appendix A: C^*-algebras and von Neumann algebras[28]

A $*$-*algebra* is an algebra closed with respect to an involution $\mathcal{A} \ni A \mapsto A^* \in \mathcal{A}$ satisfying $(A^*)^* = A$, $(A+B)^* = A^* + B^*$, $(cA)^* = \bar{c}A^*$ and $(AB)^* = B^*A^*$ for all $A, B \in \mathcal{A}$ and all complex c (where the overbar denotes the complex conjugate). A C^*-*algebra* $\mathcal{A}$ is a $*$-algebra equipped with a norm, satisfying $\|A^*A\| = \|A\|^2$ and $\|AB\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathcal{A}$, and is complete in the topology induced by that norm. A *state* ω on a C^*-algebra $\mathcal{A}$ is a positive linear map $\omega\colon \mathcal{A} \to \mathbb{C}$, where positivity means that $\omega(A^*A) \geq 0$ for any $A \in \mathcal{A}$. A state is *mixed* if it can be written as a nontrivial linear combination of other states, otherwise it is *pure*. A state ω on $\mathcal{A}$ is *faithful* just in case $\omega(A^*A) > 0$ for any nonzero $A \in \mathcal{A}$.

A *von Neumann algebra* $\mathfrak{M}$ is a particular kind of C^*-*algebra*—a concrete C^*-algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$ and closed in the weak topology of this space. A sequence of bounded operators $O_1, O_2, \ldots$ converges in the *weak topology* to O just in case $\langle\psi_1|O_j|\psi_2\rangle$ converges to $\langle\psi_1|O|\psi_2\rangle$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$. By von Neumann's Double Commutant Theorem, requiring that $\mathfrak{M}$ is weakly closed is equivalent to requiring that $\mathfrak{M} = \mathfrak{M}'' := (\mathfrak{M}')'$, where $'$ denotes the commutant (cf. p. 272). The *center* $\mathcal{Z}(\mathfrak{M})$ of a von Neumann algebra is $\mathfrak{M} \cap \mathfrak{M}'$. $\mathfrak{M}$ is a *factor* algebra just in case its center is trivial, i.e. consists of $\mathbb{C}I$. Appendix B reviews the classification of factor von Neumann algebras.

A *representation* of a C^*-algebra $\mathcal{A}$ is a homeomorphism $\pi\colon \mathcal{A} \to \mathfrak{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. The representation π is *irreducible* just in case $\pi(\mathcal{A})$ does not leave invariant a nontrivial subspace of $\mathcal{H}$. Any state ω on a C^*-algebra $\mathcal{A}$ determines a triple $(\pi_\omega, \mathcal{H}_\omega, |\Omega_\omega\rangle)$ such that the representation $\pi_\omega\colon \mathcal{A} \to \mathfrak{B}(\mathcal{H}_\omega)$ is cyclic with respect to the vector $|\Omega_\omega\rangle \in \mathcal{H}_\omega$ (i.e. $\{\pi_\omega(\mathcal{A})\,|\Omega_\omega\rangle\}$ is dense in $\mathcal{H}_\omega$), and $\omega(A) = \langle\Omega_\omega \mid \pi_\omega(A) \mid \Omega_\omega\rangle$ for all $A \in \mathcal{A}$. This GNS-representation is unique up to unitary equivalence. Since every representation of $\mathcal{A}$ is a direct

[28]One standard reference is Kadison & Ringrose 1997a, 1997b.

sum of cyclic representations, the GNS-representations can be regarded as the fundamental ones. A basic result about GNS-representations is that the state ω is pure just in case π_ω is irreducible. The von Neumann algebra associated with a representation π of $\mathcal{A}$ is $\big(\pi(\mathcal{A})\big)'' := \big(\pi(\mathcal{A})'\big)'$. Thus, if π is irreducible (as is the case with the GNS-representation induced by a pure state), $\pi(\mathcal{A})' = \mathbb{C}I$ (because if there were some nontrivial subspace invariant under $\pi(\mathcal{A})$, the projection operator for that subspace would belong to $\pi(\mathcal{A})'$), and $\pi(\mathcal{A})'' = \mathfrak{B}(\mathcal{H})$.

A state ω on a von Neumann algebra $\mathfrak{M}$ acting on $\mathcal{H}$ is *normal* just in case there is a density operator W on $\mathcal{H}$ such that $\omega(A) = \mathrm{Tr}(WA)$ for all $A \in \mathfrak{M}$. Equivalently, a normal state is a *completely additive* state, i.e. $\omega\big(\sum_\alpha E_\alpha\big) = \sum_\alpha \omega(E_\alpha)$ for any set $\{E_\alpha\}$ of pairwise orthogonal projectors in $\mathfrak{M}$. If $\mathfrak{M}$ acts on a separable Hilbert space, countable additivity (i.e. the index α runs over a countable set) suffices for normality. A *vector state* ω for a von Neumann algebra $\mathfrak{M}$ acting on $\mathcal{H}$ is a state such that there is a $|\psi\rangle \in \mathcal{H}$ where $\omega(A) = \langle\psi|A\psi\rangle$ for all $A \in \mathfrak{M}$. Vector states are normal, but the converse is not true for all types of von Neumann algebras—it is true for Type III but false for Type I.

Appendix B: Type classification of factor von Neumann algebras

The *range* of a projection E in a von Neumann algebra $\mathfrak{M}$ acting on a Hilbert space $\mathcal{H}$ is the linear span of $\{\,|\psi\rangle \in \mathcal{H}\colon\ E|\psi\rangle = |\psi\rangle\,\}$. Thus the range of E is a closed subspace of $\mathcal{H}$ (cf. Kadison & Ringrose 1997a, Prop. 2.5.1). Two projections E and F in $\mathfrak{M}$ are *equivalent* (written $E \sim F$) just in case their ranges are isometrically embeddable into one another, *by an isometry that is an element of* $\mathfrak{M}$. Equivalence so construed is manifestly relative to $\mathfrak{M}$. When E's range is a subspace of F's range (written $E \leq F$), E is a *subprojection of F*. Equivalent criteria are that $FE = EF = E$ and that $\big|E\,|\psi\rangle\big| \leq \big|F\,|\psi\rangle\big|$ for all $|\psi\rangle \in \mathcal{H}$. We use the subprojection relation to define the *weaker than* relation $\preceq$, which imposes a partial order on projections in a von Neumann algebra: E is *weaker than* F if and only if E is equivalent to a subprojection of F. Because $\preceq$ is a partial order, $E \preceq F$ and $F \preceq E$ together imply that $E \sim F$.

A projection $E \in \mathfrak{M}$ is *infinite* if and only if there's some projection $E_0 \in \mathfrak{M}$ such that $E_0 < E$ and $E \sim E_0$. In this case, E_0's range is both a proper subset of, and isometrically embeddable into, E's range. $E \in \mathfrak{M}$ is *finite* if and only if it is not infinite. A nonzero projection $E \in \mathfrak{M}$ is *minimal* if and only if E's only subprojections are 0 and E itself. It follows that minimal projections are finite. A general von Neumann algebra $\mathfrak{M}$ need not be even a direct sum of factors. In the general case, the structural analogue of a minimal projection is an *abelian* projection. A nonzero projection $E \in \mathfrak{M}$ is *abelian* if and only if the von Neumann algebra $E\mathfrak{M}E$ (in which E serves as the identity), acting on the Hilbert space $E\mathcal{H}$, is abelian. We have minimal $\Rightarrow$ abelian $\Rightarrow$ finite, but in general the arrows cannot be reversed.

The Murray–von Neumann classification of von Neumann algebras applies in the first instance to factor algebras; on such algebras, the weaker-than relation $\preceq$ imposes a total order (see Kadison & Ringrose 1997b, Prop. 6.2.6).

Type I: *Type-I factors contain minimal projections, which are therefore also abelian and finite.*

The algebras $\mathfrak{B}(\mathcal{H})$ of bounded operators on a separable Hilbert space—these include most observable-algebras familiar from discussions of nonrelativistic quantum mechanic—are Type-I factors, and each Type-I factor is isomorphic to some $\mathfrak{B}(\mathcal{H})$. For $\mathcal{H}$ n-dimensional, $\mathfrak{B}(\mathcal{H})$ is a factor of Type I_n.

Type-II and -III factors are decidedly more exotic.

Type II: *Type-II factors contain no minimal projections, but do contain* (*nonzero*) *finite projections.*

Indeed, in a sense that can be made precise (Sunder 1986, Sec. 1.3), Type-II factors have projections whose ranges are subspaces of *fractional* dimension. In a factor of Type II_1, the identity operator is finite; in a factor of Type II_∞, the identity operator is infinite.

Type III: *Type-III factors have no* (*nonzero*) *finite projections and so neither minimal nor abelian projections. All their projections are infinite and therefore equivalent.*[29]

Type-III factors also have subtypes, but it takes the resources of modular theory to characterize them (see Sunder 1986, Ch. 3). For non-factor algebras the classification is a bit more subtle. A von Neumann algebra $\mathfrak{M}$ is Type I if it contains an abelian projector with central carrier I. (The *central carrier* C_A of $A \in \mathfrak{M}$ is the intersection of all projectors $E \in \mathcal{Z}(\mathfrak{M})$ such that $EA = A$.) $\mathfrak{M}$ is Type II if it contains no nonzero abelian projectors but does contain finite projectors with central carrier I. And it is Type III if it contains no nonzero finite projections.

Appendix C: Lattices

The set $\mathcal{P}(\mathfrak{M})$ of projection operators in the von Neumann algebra $\mathfrak{M}$ is partially ordered by the relation $\leq$ of subspace inclusion. This partial order enables us to define for each pair of elements $E, F \in \mathcal{P}(\mathfrak{M})$, their *greatest lower bound* (aka *meet*) $E \cap F$ as the projection whose range is the largest closed subspace of $\mathcal{H}$ that is contained in both E and F; and their *least upper bound* (aka *join*) $E \cup F$ as the projection whose range is the smallest closed subspace of $\mathcal{H}$ that contains both E and F. A *lattice* is a partially ordered set every pair of elements of which has both a least upper bound and a greatest lower bound. Thus the foregoing definitions render $\mathcal{P}(\mathfrak{M})$ a lattice.

A lattice S has a zero element 0 such that $0 \leq a$ for all $a \in S$ and a unit element 1 such that $a \leq 1$ for all $a \in S$. The zero operator (the projection operator

[29] Cf. Kadison & Ringrose 1997b, Cor. 6.3.5.

for the null subspace) is the zero element of $\mathcal{P}(\mathfrak{M})$ and the identity operator I is the unit element. The *complement* of an element a of a lattice S is an element $a' \in S$ such that $a' \cup a = 1$ and $a' \cap a = 0$. A lattice is *complemented* if each of its elements has a complement. It is *orthocomplemented* if these complements obey

$$a'' = a \quad \text{and} \quad a \leq b \text{ if and only if } b' \leq a'.$$

$\mathcal{P}(\mathfrak{M})$ is an orthocomplemented lattice, with the complement $E^{\perp}$ of $E \in \mathcal{P}(\mathfrak{M})$ supplied by the range.

All a, b, c in a *distributive* lattice S satisfy the distributive laws

$$\begin{aligned} a \cup (b \cap c) &= (a \cup b) \cap (a \cup c), \\ a \cap (b \cup c) &= (a \cap b) \cup (a \cap c). \end{aligned}$$

Notoriously, $\mathcal{P}(\mathfrak{M})$ need not be distributive. But in the special case that $\mathfrak{M}$ is abelian, $\mathcal{P}(\mathfrak{M})$ is distributive. Indeed, it's a *Boolean lattice* (aka a *Boolean algebra*), that is, it is a distributive complemented lattice. The simplest Boolean lattice is the set $\{0, 1\}$, where each element is the other's complement, and meet and join correspond to set-theoretic intersection and union respectively. Call this lattice B_2. Notice that B_2's elements can be put into one–one correspondence with the truth values *false* (0) and *true* (1). A *Boolean homomorphism* between Boolean lattices B and B_2 is a map $h \colon B \to B_2$ preserving the Boolean operations:

$$\begin{aligned} h(a \cup b) &= h(a) \cup h(b), \\ h(a \cap b) &= h(a) \cap h(b), \\ h(a') &= h(a)', \\ h(0) &= 0, \\ h(1) &= 1. \end{aligned}$$

(*N.B.*: the second and third—equivalently the second and fourth—are sufficient to define a Boolean homomorphism; the remaining properties are consequences.)

Construing B as a lattice of propositions, we can construe lattice operations—meet ($\cap$), join ($\cup$), and complement ($'$)—as logical operations—disjunction ($\vee$), conjunction (&), and negation ($\sim$), respectively. Given this construal, a two-valued homomorphism $h \colon B \to B_2$ on a Boolean lattice B is seen to be a *truth valuation on B respecting the classical truth tables* for disjunction, conjunction, and negation.