Coupled differential equations

4/1/2018

Example:

$$\frac{dy_1}{dx} = a_{11}y_1 + a_{12}y_2 + b_1(x)$$

$$\frac{dy_2}{dx} = a_{21}y_1 + a_{22}y_2 + b_2(x)$$

Consider the case with $b_1 = b_2 = 0$

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{Ay} \Rightarrow \boldsymbol{y} = e^{\boldsymbol{A}}\boldsymbol{y}_0$$

One way to address this sort of problem, is to find the eigenvalues of the matrix and transform to the diagonal representation

Let $\boldsymbol{y} = \boldsymbol{Pz}$ ← change of basis set

$$\frac{d\boldsymbol{y}}{dx} = \boldsymbol{P}\frac{d\boldsymbol{z}}{dx} = \boldsymbol{APz}$$

with the proper choice of ***P***, we can diagonalize ***A***

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\boldsymbol{P}^{-1}\frac{d\boldsymbol{y}}{dx} = \frac{d\boldsymbol{z}}{dx} = \boldsymbol{P}^{-1}\boldsymbol{APz} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\frac{dz_1}{dx} = \lambda_1 z_1 \rightarrow z_1 = c_1 e^{\lambda_1 x}$$

$$\frac{dz_2}{dx} = \lambda_2 z_2 \rightarrow z_2 = c_2 e^{\lambda_2 x}$$

Note: ***A*** might not be Hermitian (or even symmetric). In such a case $\boldsymbol{P}^{-1}$ can be determined using cofactors and the determinant (See updated notes on Matrices)

to find *y* multiply ***z*** by ***P***

$$\boldsymbol{y} = \boldsymbol{Pz}$$

Example $\dfrac{dy_1}{dx} = y_1 + 2y_2$

$$\frac{dy_2}{dx} = 2y_1 + y_2$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$$

$$1-\lambda = \pm 2 \Rightarrow \lambda = -1, \quad +3$$

$$\left.\begin{aligned}(1+1)a_1 + 2a_2 = 0 \Rightarrow a_1 = -a_2 \\ (1-3)a_1 + 2a_2 = 0 \Rightarrow a_1 = a_2\end{aligned}\right. \Rightarrow P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$z_1 = c_1 e^{-x}, \quad z_2 = c_2 e^{3x}$$

$$\boldsymbol{y} = \boldsymbol{Pz}$$

$$y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{-x} \\ c_2 e^{3x} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} c_1 e^{-x} + c_2 e^{3x} \\ -c_1 e^{-x} + c_2 e^{3x} \end{pmatrix}$$

$$y_1 = \frac{1}{\sqrt{2}}\left(c_1 e^{-x} + c_2 e^{3x}\right)$$

$$\frac{dy_1}{dt} = \frac{1}{\sqrt{2}}\left(-c_1 e^{-x} + 3c_2 e^{3x}\right)$$

$$y_1 + 2y_2 = c_1 e^{-x} + c_2 e^{3x} + 2\left(-c_1 e^{-x} + c_2 e^{3x}\right) = -c_1 e^{-x} + 3c_2 e^{3x}$$

Consistency check

Higher order differential equations can be converted to systems of first-order equations

Consider $m\frac{d^2x}{dt^2}+kx=0$

$$\frac{d^2x}{dt^2}+\frac{kx}{m}=0$$

Let $\frac{dx}{dt}=v$

Then

$$\frac{dv}{dt}+\frac{kx}{m}=0$$

$$\frac{dx}{dt}-v=0$$

$$\frac{d}{dt}\begin{pmatrix} v \\ x \end{pmatrix}=\begin{pmatrix} 0 & k/m \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v \\ x \end{pmatrix}$$

Thus, we have converted 2nd-order differential equation to two coupled first-order equations

Initial conditions

$x(0)=x_0 \qquad v(0)=v_0$

Can solve using matrix techniques

Can also solve numerically using Euler or Runge Kutta methods

Consider Euler

$$x(1) = x(0) - v(0)\Delta t$$

$$v(1) = v(0) - \frac{k}{m}x(0)\Delta t$$

Note: could also use x_1 at this point since the first equation gives a new value of x.

$$\downarrow$$

$$x(2) = x(1) - v(1)\Delta t$$

$$v(2) = v(1) - \frac{k}{m}x(1)\Delta t$$

$$\vdots$$

See posted handout as to how to solve numerically with Mathematica

Power series solutions to differential equations

E.g. $\frac{dy}{dt} = ky$ suppose we didn't know how to solve this

Write $y = a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{n=0}^{\infty} a_n t^n$

$$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + \ldots = k\left[a_0 + a_1 t + a_2 t^2 + \ldots\right] \Rightarrow$$

$$(a_1 - ka_0) + (2a_2 - ka_1)t + (3a_3 - ka_2)t^2 + \ldots = 0$$

For this to be satisfied in the general case, each term in parentheses must be 0.

$$\Rightarrow a_1 = ka_0$$

$$a_2 = \frac{1}{2}ka_1 = \frac{1}{2}k^2 a_0$$

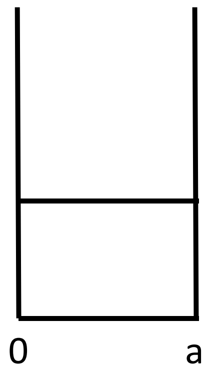
$$a_3 = \frac{1}{3}ka_2 = \frac{1}{3 \bullet 2}k^2 a_1 = \frac{1}{3 \bullet 2}k^3 a_0$$

The eq. expressing the high order coefficients in terms of lower order coefficients is called a recursion relation

$$y = a_0\left[1 + kt + \frac{1}{2}(kt)^2 + \frac{1}{6}(kt)^3 \ldots\right]$$

$$y = a_0 e^{kt}$$

$$t = 0 \Rightarrow a_0 = y_0 \Rightarrow y = y_0 e^{kt}$$



Now lets treat the quantum mechanical particle-in-the-box problem

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\Psi = E\psi, \quad 0 \le x \le a$$

Inside the box

$$\frac{d^2}{dx^2}\psi = -\alpha\psi, \quad \alpha = \frac{2mE}{\hbar^2}$$

$$\frac{d^2\psi}{dx^2} + \alpha\psi = 0$$

$$\left[2a_2 + 3 \bullet 2a_3 x + 4 \bullet 3x^2 + \ldots\right] + \alpha\left[a_0 + a_1 x + a_2 x^2 + \ldots\right]$$

$$2a_2 + \alpha a_0 = 0 \rightarrow a_2 = -\frac{\alpha}{2}a_0$$

$$6a_3 + \alpha a_1 = 0 \rightarrow a_3 = -\frac{\alpha}{6}a_1$$

$$12a_4 + \alpha a_2 = 0 \rightarrow a_4 = -\frac{\alpha}{4 \bullet 3}a_2 = \frac{\alpha^2}{4 \bullet 3 \bullet 2}a_0$$

$$20a_5 + \alpha a_3 = 0 \rightarrow a_5 = -\frac{\alpha}{5 \bullet 4}a_3 = \frac{\alpha^2}{5 \bullet 4 \bullet 3 \bullet 2}a_1$$

recursion relations

$$\psi = a_0\left[1-\frac{\alpha}{2}x^2+\frac{\alpha^2}{4\cdot3\cdot2}x^4+\ldots\right]+a_1\left[x-\frac{\alpha}{3\cdot2}x^3+\frac{\alpha^2}{5\cdot4\cdot3\cdot2}x^5-\ldots\right]$$

We also know $\psi(0)=0 \Rightarrow a_0=0$

Note $\sin(kx)=kx-\frac{(kx)^3}{3!}+\frac{(kx)^5}{5!}-\ldots$

$$\frac{1}{k}\sin(kx)=x-\frac{(k^2x^3)}{3!}+\frac{k^4x^5}{5!}-\ldots$$

$$\psi=\frac{1}{\sqrt{\alpha}}\left[\sqrt{\alpha}x-\frac{\alpha^{3/2}x^3}{3!}+\frac{\alpha^{5/2}x^5}{5!}-\ldots\right]=\frac{1}{k}\sin(kx)$$

| let $k^2=\alpha$

Apply boundary condition at *x* = *a*

$$\sqrt{\alpha}a=n\pi \Rightarrow \sqrt{\alpha}=\frac{n\pi}{a}$$

$$\sin(kx)=\sin\left(\frac{n\pi x}{a}\right)$$

$$\alpha=\frac{2mE}{\hbar^2}=\frac{n^2\pi^2}{a^2}$$

but

$$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}=\frac{n^2h^2}{8ma^2},\quad n=1,2,3,\ldots$$

$n=1:\quad E_1,\psi_1$

$n=2:\quad E_2,\psi_2$

etc.

Now consider the quantum harmonic oscillator

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}kx^2\psi=E\psi$$

$$-\frac{d^2\psi}{dx^2}+\frac{mk}{\hbar^2}x^2\psi=\frac{2mE}{\hbar^2}\psi$$

$$-\frac{d^2\psi}{dx^2}+\alpha x^2=\varepsilon\psi$$

try $\psi=e^{-ax^2}$

$$\psi'=-2axe^{-ax^2}$$

$$\psi''=\left(-2a+4a^2x^2\right)e^{-ax^2}\Rightarrow$$

$$-\left(-2a+4a^2x^2\right)+\alpha x^2=\varepsilon$$

$$\varepsilon=2a,\quad \alpha=4a^2$$

$$+\frac{2mE}{\hbar^2}=2a\,,\ E=\frac{\hbar^2 a}{m}$$

$$a=\frac{\sqrt{\alpha}}{2}=\frac{\sqrt{mk}}{2\hbar}$$

$$E=\frac{\hbar^2}{m}\frac{\sqrt{mk}}{2\hbar}=\sqrt{\frac{k}{m}}\frac{\hbar}{2}=\frac{1}{2}\hbar\omega=\frac{1}{2}h\upsilon$$

We will see later how we can solve for the excited state.

What would be reasonable guess for the wave function of the first excited state?

For a general solution, we could try $\psi=e^{-ax^2}[a_0+a_1x+...]$

Non-linear differential equations

Logistic equation

$$\frac{dA}{dt} = \varepsilon A - \sigma A^2, \qquad A = \text{population}$$

εA → population growth

$-\sigma A^2$ → intraspecies competition for resources

In the absence of the σA^2 term

$$\frac{dA}{dt} = \varepsilon A \Rightarrow A = A_0 e^{\varepsilon t}$$

Critical points (equilibrium points)

$$\frac{dA}{dt} = \varepsilon A - \sigma A^2,$$

$$\varepsilon A - \sigma A^2 = 0$$

$$\Rightarrow A = 0,$$

$$or\ A = \varepsilon / \sigma$$

if $A \neq 0 \ \ or\ \varepsilon / \sigma$

$$\frac{dA}{\varepsilon A - \sigma A^2} = dt$$

$$\left(\frac{1}{A} + \frac{-1}{A - \frac{\varepsilon}{\sigma}} \right) dA = \varepsilon dt$$

$$\ell n(A) - \ell n\left(A - \frac{\varepsilon}{\sigma} \right) = \varepsilon t$$

$$\ell n\left(\frac{-\varepsilon / \sigma + A}{A} \right) = -\varepsilon t$$

$$\left(\frac{1}{A} + \frac{-1}{A - \frac{\varepsilon}{\sigma}} \right) = \frac{(A - \frac{\varepsilon}{\sigma}) - A}{A\left(A - \frac{\varepsilon}{\sigma} \right)}$$

$$= \frac{-\frac{\varepsilon}{\sigma}}{A^2 - \frac{\varepsilon}{\sigma} A} = \frac{\varepsilon}{-A^2\sigma + \varepsilon A}$$

$$\frac{-\varepsilon/\sigma + A}{A} = ce^{-\varepsilon t}$$

$$-\varepsilon/\sigma + A = Ace^{-\varepsilon t}$$

$$A\left(ce^{-\varepsilon t} - 1\right) = -\varepsilon/\sigma$$

$$A = \frac{\varepsilon/\sigma}{1 - ce^{-\varepsilon t}}$$

if $A(0) = A_0$

$$A_0 = \frac{\varepsilon/\sigma}{1-c} \rightarrow$$

$$(1-c)A_0 = \varepsilon/\sigma$$

$$cA_0 = A_0 - \varepsilon/\sigma$$

$$c = \frac{A_0 - \varepsilon/\sigma}{A_0}$$

$A_0 = 0 \Rightarrow A = 0$

$A_0 = \varepsilon/\sigma \Rightarrow A = \varepsilon/\sigma$ carrying capacity

With either of these initial conditions, the population does not change with time

$$A = \frac{\varepsilon}{\sigma + \left[\frac{\varepsilon - \sigma A_0}{A_0}\right] e^{-\varepsilon t}}$$

As $t \to \infty$ $A \to \varepsilon / \sigma =$ carrying capacity (if $A_0 \neq 0$)

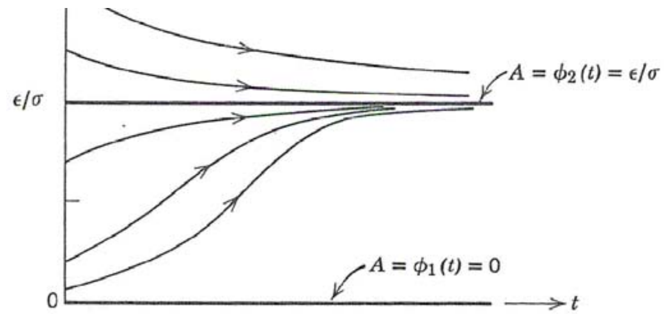


FIGURE 9.1 $dA/dt = \epsilon A - \sigma A^2$, $\epsilon > 0$, $\sigma > 0$.

From Boyce and DiPrima

Limiting behavior can often be established without solving the differential equation

Consider plot of *dA/dt vs. A,* $\varepsilon > 0$, $\sigma > 0$

if *dA/dt +*, *A* increases toward ε / σ
if *dA/dt -*, A decreases toward ε / σ

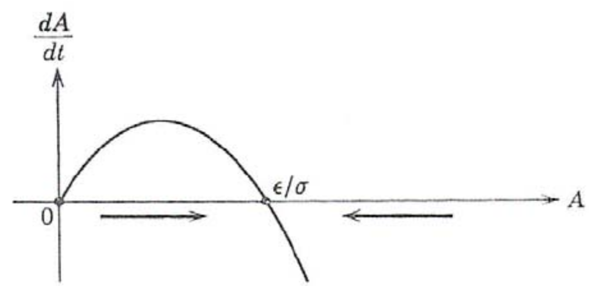


FIGURE 9.2 $dA/dt = \epsilon A - \sigma A^2$, $\epsilon > 0$, $\sigma > 0$.

From Boyce and DiPrima

A = 0 is an unstable critical point, since if A_0 is only infinitesimally >0, the system evolves to ε / σ
$A = \varepsilon / \sigma$ is an asymptotically stable critical point

Now consider $\varepsilon < 0$, $\sigma < 0$

any start with $0 < A_0 < \varepsilon / \sigma$ evolves to 0
any start with $A_0 > \varepsilon / \sigma$ grows without bound

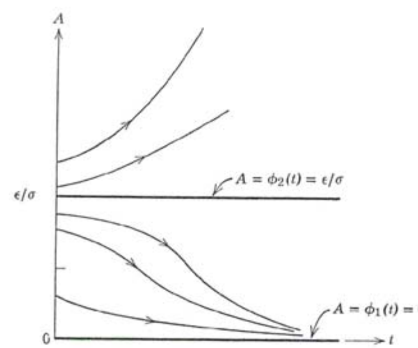


FIGURE 9.5 $dA/dt = \epsilon A - \sigma A^2$, $\epsilon < 0$, $\sigma < 0$.

From Boyce and DiPrima

Coupled Non-linear Differential Equations

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

Example

$$\frac{dx}{dt} = \varepsilon_1 x - \sigma_1 x^2 - \alpha_1 xy = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$\frac{dy}{dt} = \varepsilon_2 y - \sigma_2 y^2 - \alpha_2 xy = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x)$$

$x = 0,$ 2nd equation $\varepsilon_2 - \sigma_2 y = 0$

$y = 0,$ 1st equation $\varepsilon_1 - \sigma_1 x = 0$

critical points $(x, y) = (0,0), \quad (0, \varepsilon_2 / \sigma_2), \quad (\varepsilon_1 / \sigma_1, 0)$

also constant solution if $\varepsilon_1 - \sigma_1 x - \alpha_1 y$ and $\varepsilon_2 - \sigma_2 y - \alpha_2 x$ intersect.

With increasing time:

x increases (decreases) if $\varepsilon_1 - \sigma_1 x - \alpha_1 y \quad > (<) 0$

y increases (decreases) if $\varepsilon_2 - \sigma_2 y - \alpha_2 x \quad > (<) 0$

Example

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right)$$

critical points $(0,0),\quad (0,2),\quad (1,0),\quad \left(\frac{1}{2},\frac{1}{2}\right)$

In many cases, one can learn about the behavior of the solutions
in the vicinity of the various critical points by linearizing about each critical point

E.g., (0,0)

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = \frac{1}{2}y$$
$$x = e^{t}$$
$$y = e^{0.5t}$$

This is an unstable critical point

$$\frac{dx}{dt} = x(1 - x - y)$$

$$\frac{dy}{dt} = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right)$$

Now consider $(x, y) = (1, 0)$

Let $x = u + 1, \quad y = v$

$$\frac{du}{dt} = -u - v - u^2 - uv$$

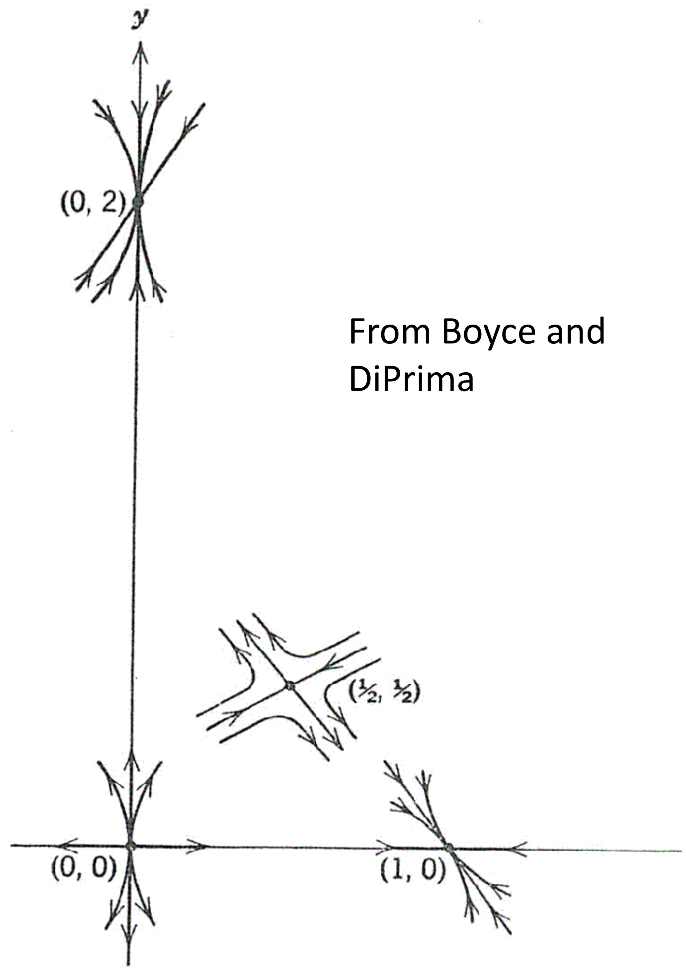
$$\frac{dv}{dt} = -\frac{1}{4}v - \frac{1}{4}v^2 - \frac{3}{4}uv$$

Linearize

$$\frac{du}{dt} = -u - v, \quad \frac{dv}{dt} = -\frac{1}{4}v$$

$$u = -\frac{4}{3}Ae^{-t/4} + Be^{-t}, \quad v = Ae^{-t/4}$$

This is an asymptotically stable point



From Boyce and DiPrima

FIGURE 9.27a

Specialize to predator-prey problem

$$\frac{dR}{dt} = aR - \alpha RF \quad \Big| \quad \text{Rabbit}$$

$$\frac{dF}{dt} = -cF + \gamma RF \quad \Big| \quad \text{Fox}$$

assume all constants are positive

Find the critical points

$$R\left(a - \alpha F\right) = 0$$
$$F\left(-c + \gamma R\right) = 0$$

critical points $\left(R, F\right) = \left(0, 0\right), \quad \left(\frac{c}{\gamma}, \frac{a}{\alpha}\right)$

For the critical point $(R,F)=\left(\frac{c}{\gamma},\frac{a}{\alpha}\right)$

$$R=\frac{c}{\gamma}+r \quad F=\frac{a}{\alpha}+f$$

$$\frac{dR}{dt}=aR-\alpha RF,\quad \frac{dF}{dt}=-cF+\gamma RF$$

$$\frac{dr}{dt}=a\left(\frac{c}{\gamma}+r\right)-\alpha\left(\frac{c}{\gamma}+r\right)\left(\frac{a}{\alpha}+f\right)$$

$$\frac{dr}{dt}=\frac{ac}{\gamma}+ar-\alpha\left(\frac{ca}{\gamma\alpha}+\frac{c}{\gamma}f+\frac{a}{\alpha}r+rf\right)$$

$$\frac{dr}{dt}=\frac{ac}{\gamma}+ar-\frac{ac}{\gamma}-\frac{ac}{\gamma}f-ar-\alpha rf$$

$$=-\frac{ac}{\gamma}f-\alpha rf$$

For (r,f) the critical pointed is shifted to (0,0)

$$\frac{dF}{dt} = -cF + \gamma RF$$

$$\frac{df}{dt} = -c\left(\frac{a}{\alpha} + f\right) + \gamma\left(\frac{c}{\gamma} + r\right)\left(\frac{a}{\alpha} + f\right)$$

$$= -\frac{ca}{\alpha} - cf + (c + \gamma r)\left(\frac{a}{\alpha} + f\right)$$

$$= -\frac{ca}{\alpha} - cf + \frac{ca}{\alpha} + cf + \frac{\gamma a}{\alpha} r + \gamma rf$$

$$= \frac{\gamma a}{\alpha} r + \gamma rf$$

The linearized equations become

$$\frac{dr}{dt} = -\frac{\alpha c}{\gamma} f; \quad \frac{df}{dt} = \frac{\gamma a}{\alpha} r$$

$$\frac{d}{dt}\begin{pmatrix} r \\ f \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\alpha c}{\gamma} \\ \frac{\gamma a}{\alpha} & 0 \end{pmatrix}\begin{pmatrix} r \\ f \end{pmatrix}$$

$$\lambda_1 = i\sqrt{ac}, \quad \lambda_1 = -i\sqrt{ac}$$

So

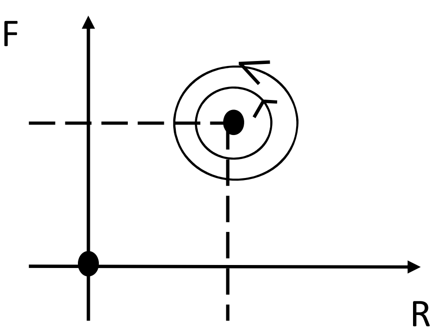
$$r = c_1 e^{i\sqrt{ac}t} + c_2 e^{-i\sqrt{ac}t}$$

$$f = c_3 e^{i\sqrt{ac}t} + c_4 e^{-i\sqrt{ac}t}$$

Eigenvalues of matrix

$$+\lambda^2 + \frac{\alpha c}{\gamma}\frac{\gamma a}{\alpha} = 0$$

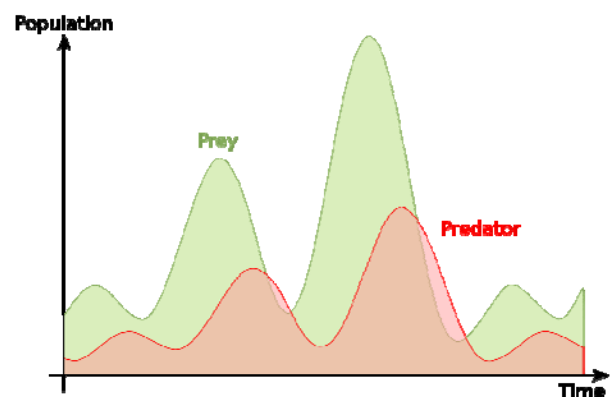
$$\lambda = +i\sqrt{ac}$$



Both populations oscillate. If we solve for the coefficients we will see that, the rabbit population grows, which causes the fox population to grow.

Once the fox population gets high, the rabbit population begins to fall, followed by a drop in the fox population.

This cycle repeats itself.



From Wikipedia