Homework 3 Answer Key

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1 Question 1

Is $\sin(\frac{\pi x}{a})$ an eigenfunction of the momentum operator, $\hat{p}$, over the interval $[0, a]$?

$$\hat{p}_x = -i\hbar\frac{d}{dx}$$

$$\hat{p}_x(f(x)) = \lambda(f(x))$$

$$-i\hbar\frac{d}{dx}\sin(\frac{\pi x}{a}) = -i\hbar\frac{\pi}{a}\cos(\frac{\pi x}{a})$$

Thus, not an eigenfunction.

1.1 What values can you get from individual measurements?

The values you can receive are the values from the component eigenfunctions of the momentum operator that create the superposition. Knowing that $\sin(\frac{\pi x}{a}) = \frac{1}{2i}(e^{\frac{i\pi x}{a}} - e^{\frac{-i\pi x}{a}})$ you can find the exact values that can be measured. This is also a the wavefunction written as a linear combination of eigen functions of the momentum operator.

Perform the operation on each component eigenfunction individually:

$$-i\hbar\frac{d}{dx}(e^{\frac{i\pi x}{a}}) = -i\hbar(\frac{i\pi}{a})(e^{\frac{i\pi x}{a}}) = \frac{\hbar\pi}{a}(e^{\frac{i\pi x}{a}})$$

and

$$-i\hbar\frac{d}{dx}(e^{-\frac{i\pi x}{a}}) = i\hbar(\frac{i\pi}{a})(e^{-\frac{i\pi x}{a}}) = -\frac{\hbar\pi}{a}(e^{-\frac{i\pi x}{a}})$$

Thus, the two eigenvalues are $\pm\frac{\hbar\pi}{a}$.

1.2 What is the average momentum?

$$\frac{\int_{-\infty}^{\infty}\Psi^*\hat{p}_x\Psi dx}{\int_{-\infty}^{\infty}\Psi^*\Psi dx}$$

This equation can be simplified by having a normalized wavefunction since the denominator will be equal to one. Normalization:

$$N^2\int_0^a \sin^2(\frac{\pi x}{a})dx = 1$$

$$\frac{N^2}{2}\int_0^a (1-\cos(\frac{2\pi x}{a})dx$$

$$\frac{N^2}{2}(x-\frac{a}{2\pi}\sin(\frac{2\pi x}{a})\Big|_0^a = 1$$

$$\frac{N^2}{2}(a) = 1$$

$$N = \sqrt{\frac{2}{a}}$$

Using the normalized wave function, solve for the average/expected value.

$$\frac{-i\hbar\int_0^a(\sqrt{\frac{2}{a}})\sin(\frac{\pi x}{a})\frac{d}{dx}(\sqrt{\frac{2}{a}})\sin(\frac{\pi x}{a})dx}{\int_0^a(\frac{2}{a})\sin^2(\frac{\pi x}{a})dx = 1}$$

$$(\frac{\pi}{a})(-i\hbar)(\frac{2}{a})\int_0^a \sin(\frac{\pi x}{a})\cos(\frac{\pi x}{a})dx$$

Table of integrals shows that $(\frac{\pi}{a})(-i\hbar)\int_0^a \sin(\frac{\pi x}{a})\cos(\frac{\pi x}{a})dx = 0$ so the average momentum is zero.

1.3 What is the average position, $\hat{x}$?

$$\frac{\int_0^a(\sqrt{\frac{2}{a}})\sin(\frac{\pi x}{a})x(\sqrt{\frac{2}{a}})\sin(\frac{\pi x}{a})dx}{\int_0^a(\frac{2}{a})\sin^2(\frac{\pi x}{a})dx = 1}$$

$$(\frac{2}{a})\int_0^a x\sin^2(\frac{\pi x}{a})dx$$

$$\frac{2}{a}\times\frac{1}{2}\left[\int_0^a x(1-\cos(\frac{2\pi x}{a}))dx\right]$$

$$\frac{1}{a}\left[\int_0^a xdx - \int_0^a x\cos(\frac{2\pi x}{a})dx\right]$$

Integration by parts and solving the first integral:

$$\frac{1}{a}\left[\left(\frac{x^2}{2}\right)\Big|_0^a - \left(x\frac{a}{2\pi}\sin(\frac{2\pi x}{a})\Big|_0^a - \int_0^a \frac{a}{2\pi}\sin(\frac{2\pi x}{a})dx\right)\right]$$

$$\frac{a}{2}-\frac{1}{a}\left(\frac{a^2}{2\pi}\times 0 + \frac{a}{2\pi}\cos(\frac{2\pi x}{a})\frac{a}{2\pi}\Big|_0^a\right)$$

$$\frac{a}{2}-\frac{1}{a}\left(\frac{a^2}{4\pi}(1-1)\right) = \frac{a}{2}$$

2 Question 2

2.1 Normalization of the wavefunction

$$N^2 \int_0^a x^2(a-x)^2 dx = 1$$

$$N^2 \int_0^a a^2x^2 - 2ax^3 + x^4 dx = 1$$

$$N^2 \left(\frac{a^5}{3} - \frac{2a^5}{4} + \frac{a^5}{5} = 1 \right)$$

$$N^2 (\frac{a^5}{30}) = 1 \Rightarrow N^2 = \frac{30}{a^5} \Rightarrow N = \sqrt{\frac{30}{a^5}}$$

Normalized wavefunction:

$$\Psi(x) = \left(\frac{30}{a^5} \right)^{\frac{1}{2}} x(a-x)$$

2.2 Average $\hat{x}$

$$\frac{\int_0^a \Psi^* \hat{x} \Psi dx}{1}$$

$$\left(\frac{30}{a^5} \right) \int_0^a x^3(a-x)^2 dx$$

$$\frac{30}{a^5} \int_0^a x^3a^2 - 2ax^4 + x^5 dx$$

$$\frac{30}{a^5} \left(\frac{x^4a^2}{4} - \frac{2ax^5}{5} + \frac{x^6}{6} \right) \Big|_0^a$$

$$\frac{30}{a^5} \left(\frac{a^6}{4} - \frac{2a^6}{5} + \frac{a^6}{6} \right) = \frac{a}{2}$$

2.3 Average $\hat{p}_x$

$$\frac{30}{a^5} \int_0^a x(a-x)(-i\hbar) \frac{d}{dx}(x(a-x)) dx$$

$$\frac{-30}{a^5} i\hbar \int_0^a (ax - x^2)(-x + a - x)) dx$$

$$\frac{-30}{a^5} i\hbar \int_0^a (ax - x^2)(a - 2x) dx \Rightarrow \frac{-30}{a^5} i\hbar \int_0^a a^2x - 2ax^2 - ax^2 + 2x^3 dx$$

$$\frac{-30}{a^5} i\hbar \left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{2x^4}{4} \right) \Big|_0^a$$

$$\frac{-30}{a^5} i\hbar \left(\frac{a^4}{2} - \frac{a^4}{3} + \frac{2x^4}{4} \right) = 0$$

$$\hat{p}_x = 0$$

2.4 Average $(\hat{p}_x)^2$

$$\frac{30}{a^5}\left(\int_0^a x(a-x)(-i\hbar)^2\frac{d^2}{dx^2}(x(a-x))dx\right)$$

$$\frac{-30}{a^5}\hbar^2\left(\int_0^a x(a-x)\frac{d}{dx}(a-2x)dx\right)$$

$$\frac{-30}{a^5}\hbar^2\int_0^a (ax-x^2)(-2)dx$$

$$\frac{60}{a^5}\hbar^2\int_0^a ax-x^2dx$$

$$\frac{60}{a^5}\hbar^2\left(\frac{ax^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^a$$

$$\frac{60}{a^5}\hbar^2\left(\frac{a^3}{2}-\frac{a^3}{3}\right)\Rightarrow\frac{10\hbar}{a^2}$$