Answer Key for Homework # 2:
Chapter 2: P2.2, P2.7, P2.12, P2.16, P2.22, P2.23
Assigned: September 3; Due: September 15

***P2.2*)** Consider a two-level system with $\varepsilon_1 = 3.10 \times 10^{-21}$J and $\varepsilon_2 = 6.10 \times 10^{-21}$J. If $g_2 = g_1$, what value of T is required to obtain $n_2/n_1 = 0.150$? What value of T is required to obtain $n_2/n_1 = 0.999$?

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left[\frac{-(\varepsilon_5 - \varepsilon_1)}{kT}\right]$$

$$\ln\left(\frac{n_2}{n_1}\right) = \ln\left(\frac{g_2}{g_1}\right) - \frac{(\varepsilon_5 - \varepsilon_1)}{kT}$$

$$\frac{1}{T} = \frac{k}{(\varepsilon_5 - \varepsilon_1)}\left[\ln\left(\frac{g_2}{g_1}\right) - \ln\left(\frac{n_2}{n_1}\right)\right]$$

$$T = \frac{(\varepsilon_5 - \varepsilon_1)}{k\left[\ln\left(\frac{g_2}{g_1}\right) - \ln\left(\frac{n_2}{n_1}\right)\right]}$$

$$\text{for } n_2/n_1 = 0.150 \quad T = \frac{3.00\times10^{-21}\,\text{J}}{1.381\times10^{-23}\,\text{J K}^{-1}\times\left[\ln(1) - \ln(0.150)\right]} = 115\,\text{K}$$

$$\text{for } n_2/n_1 = 0.999 \quad T = \frac{3.00\times10^{-21}\,\text{J}}{1.381\times10^{-23}\,\text{J K}^{-1}\times\left[\ln(1) - \ln(0.999)\right]} = 2.17\times10^{5}\,\text{K}$$

P2.7) Express the following complex numbers in the form $re^{i\theta}$.

a) $2-4i$ b) 6 c) $\dfrac{3+i}{4i}$ d) $\dfrac{8+i}{2-4i}$

In the notation $re^{i\theta}$, $r=|z|=\sqrt{a^2+b^2}$ and $\theta=\cos^{-1}\left(\dfrac{\operatorname{Re} z}{|z|}\right)$.

a) $2-4i=2\sqrt{5}\exp\left(i\cos^{-1}\dfrac{1}{\sqrt{5}}\right)=2\sqrt{5}\exp(0.352i\pi)$

b) $6=6\exp\left(i\cos^{-1}1\right)=6\exp(0)$

c) $\dfrac{3+i}{4i}=\dfrac{1}{4}-\dfrac{3i}{4}=\dfrac{\sqrt{10}}{4}\exp\left(i\cos^{-1}\dfrac{1}{\sqrt{10}}\right)=\dfrac{\sqrt{10}}{4}\exp(0.398i\pi)$

d) $\dfrac{8+i}{2-4i}=\dfrac{3}{5}+\dfrac{17i}{10}=\dfrac{\sqrt{13}}{2}\exp\left(i\cos^{-1}\dfrac{6}{5\sqrt{13}}\right)=\dfrac{\sqrt{13}}{2}\exp(0.392i\pi)$

P2.12) Determine in each of the following cases if the function in the first column is an eigenfunction of the operator in the second column. If so, what is the eigenvalue?

a) $3\cos^2\theta-1$ $\qquad\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d}{d\theta}\right)$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\left(3\cos^2\theta-1\right)}{d\theta}\right)=\frac{1}{\sin\theta}\frac{d}{d\theta}\left(-6\cos\theta\sin^2\theta\right)$$

$$=\frac{1}{\sin\theta}\left(6\sin^3\theta-12\cos^2\theta\sin\theta\right)=6\sin^2\theta-12\cos^2\theta$$

$$=6-18\cos^2\theta=-6\left(3\cos^2\theta-1\right)$$

Eigenfunction with eigenvalue –6.

b) $e^{-\frac{1}{2}x^2}$ $\qquad\qquad$ $\frac{d^2}{dx^2} - x^2$

$$\frac{d^2 e^{-\frac{1}{2}x^2}}{dx^2} - x^2 e^{-\frac{1}{2}x^2} = -e^{-\frac{1}{2}x^2}$$

Eigenfunction with eigenvalue –1.

c) $e^{-4i\phi}$ $\qquad\qquad$ $\frac{d^2}{d\phi^2}$

$$\frac{d^2 e^{-4i\phi}}{d\phi^2} = -16e^{-4i\phi}$$

Eigenfunction with eigenvalue –16.

P2.16) If two operators act on a wave function as indicated by $\hat{A}\hat{B}f(x)$, it is important to carry out the operations in succession with the first operation being that nearest to the function. Mathematically, $\hat{A}\hat{B}f(x) = \hat{A}\left(\hat{B}f(x)\right)$and $\hat{A}^2 f(x) = \hat{A}\left(\hat{A}f(x)\right)$. Evaluate the following successive operations $\hat{A}\hat{B}f(x)$. The operators $\hat{A}$ and $\hat{B}$ are listed in the first and second columns and $f(x)$is listed in the third column.

a) $$\frac{d}{dx}\left[\frac{d\left(x^2 + e^{ax^2}\right)}{dx}\right] = \frac{d}{dx}\left[2x + 2a\,x\,e^{ax^2}\right] = 2 + 4a^2x^2e^{ax^2} + 2a\,e^{ax^2}$$

b) $$\frac{\partial^2}{\partial y^2}\left[\frac{\partial\left(\cos 3y \sin^2 x\right)}{\partial x}\right] = \frac{\partial^2}{\partial y^2}\left[2\cos 3y \sin x \cos x\right] = -18\cos 3y \sin x \cos x$$

c) $$\frac{\partial}{\partial\theta}\left[\frac{\partial^2\left(\frac{\cos\phi}{\sin\theta}\right)}{\partial\phi^2}\right] = \frac{\partial}{\partial\theta}\left[-\frac{\cos\phi}{\sin\theta}\right] = \frac{\cos\phi\cos\theta}{\sin^2\theta}$$

P2.22) Show by carrying out the integration that $\sin(m\pi x/a)$ and $\cos(m\pi x/a)$, where m is an integer, are orthogonal over the interval $0 \le x \le a$. Would you get the same result if you used the interval $0 \le x \le 3a/4$? Explain your result.

$$\int_0^a \cos\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{m\pi\, x}{a}\right) d\,x = \left[\frac{a}{2m\pi}\sin^2\left(\frac{m\pi\, x}{a}\right)\right]_0^a = \frac{a}{2m\pi}\left[\sin^2(m\pi) - 0\right] = 0$$

$$\int_0^{\frac{3a}{4}} \cos\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{m\pi\, x}{a}\right) d\,x = \int_0^{\frac{3a}{4}} \cos\left(\frac{m\pi\, x}{a}\right) \sin\left(\frac{m\pi\, x}{a}\right) d\,x$$

$$= \left[\frac{a}{2m\pi}\sin^2\left(\frac{m\pi\, x}{a}\right)\right]_0^{\frac{3a}{4}} = \frac{a}{2m\pi}\left[\sin^2\left(\frac{3m\pi}{4}\right) - 0\right] \neq 0$$

except for the special case $\frac{3m}{4} = n$ where n is an integer. The length of the integration interval must be n periods (for n an integer) to make the integral zero.

P2.23) Normalize the set of functions $\phi_n(\theta) = e^{in\theta}$, $0 \le \theta \le 2\pi$. To do so, you need to multiply the functions by a so-called normalization constant N so that the integral

$$N\,N^* \int_0^{2\pi} \phi_m^*(\theta)\,\phi_n(\theta) d\theta = 1 \quad \text{for} \quad m = n$$

$N\,N^* \int_0^{2\pi} e^{-in\theta}\, e^{in\theta} d\theta = N\,N^* \int_0^{2\pi} d\theta = 2\pi\, N\,N^* = 1$ This is satisfied for $N = \frac{1}{\sqrt{2\pi}}$ and the normalized functions are $\phi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}, \quad 0 \le \theta \le 2\pi$.