or from the time derivative of the position,

$$\begin{aligned} f(t) &= m\ddot{x}(t) \\ &= m\frac{d\dot{x}(t)}{dt} \\ &= -mA\omega^2 \sin(\omega t + \phi) \\ &= -k_s A \sin(\omega t + \phi). \end{aligned}$$

2. Equalizing energies.

For the two 10-particle two-state systems of Example 3.9, suppose the total energy to be shared between the two objects is $U = U_A + U_B = 4$. What is the distribution of energies that gives the highest multiplicity?

We can write

$$W(U_A) = \left[\frac{10!}{U_A!(10-U_A)!}\right]\left[\frac{10!}{(4-U_A)!(10-4+U_A)!}\right].$$

The following table lists all the possibilities:

U	$W(U)$
0	210
1	1200
2	2025
3	1200
4	210

This shows that the highest multiplicity occurs, in this case, when the energy is divided equally between the two objects.

3. Energy conversion.

When you drink a beer, you get about 100 Cal (1 food Cal = 1 kcal). You can work this off on an exercise bicycle in about 10 minutes. If you hook your exercise bicycle to a generator, what wattage of light bulb could you light up, assuming 100% efficiency? (1 watt = 1 J s^{-1} is power, i.e., energy per unit time.)

$$(100\,\mathrm{kcal})\left(4.18\,\mathrm{J\,cal^{-1}}\right)\left(\frac{1}{10\,\mathrm{min}}\right)\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) = 697\,\frac{\mathrm{J}}{\mathrm{s}} = 697.6\,\mathrm{watts}.$$

An interesting comparison is that 1 horsepower = 746 watts.

4. Kinetic energy of a car.

How much kinetic energy does a 1700 kg car have, if it travels 100 km h^{-1}?

$$\begin{aligned}
\text{Kinetic energy} &= \frac{1}{2}mv^2 \\
&= \left(\frac{1}{2}\right)(1700\,\mathrm{kg})\left[\left(\frac{10^5\,\mathrm{m}}{\mathrm{h}}\right)\left(\frac{1\,\mathrm{h}}{3600\,\mathrm{s}}\right)\right]^2 \\
&= 655.8\,\mathrm{kJ}
\end{aligned}$$

5. Root-mean-square (RMS) velocity of a gas.

Using $(1/2)kT = (1/2)m\langle {v_x}^2\rangle$, for $T = 300\,\mathrm{K}$, compute the RMS velocity, $\langle {v_x}^2\rangle^{1/2}$, of O_2 gas.

$$\begin{aligned}
\langle {v_x}^2\rangle &= \frac{kT}{M}, \\
M &= 32\,\mathrm{g\,mol^{-1}} \times \frac{1\,\mathrm{mol}}{6.02\times 10^{23}\,\mathrm{molecules}} \times \frac{\mathrm{kg}}{1000\,\mathrm{kg}} = 5.316\times 10^{-26}\,\frac{\mathrm{kg}}{\mathrm{molecule}}, \\
\langle {v_x}^2\rangle &= 1.381\times 10^{-23}\,\mathrm{J\,K^{-1}} \times 300\,\mathrm{K} \times \frac{1}{5.316\times 10^{-26}}\,\frac{\mathrm{molecules}}{\mathrm{kg}} = 77,934.5\,\mathrm{m^2\,s^{-2}}, \\
(1\,\mathrm{J} &= 1\,\mathrm{kg\,m^2\,s^{-2}}), \\
\langle {v_x}^2\rangle^{1/2} &= 279.2\,\mathrm{m\,s^{-1}}.
\end{aligned}$$

9. Small differences of large numbers can lead to nonsense.

Using the results from Problem 8, show that the propagated error is larger than the difference itself for $f(x, y) = x - y$, with $x = 20 \pm 2$ and $y = 19 \pm 2$.

Since $(\partial f/\partial x)^2 = 1$ and $(\partial f/\partial y)^2 = 1$, we have $\varepsilon_f^2 = \varepsilon_x^2 + \varepsilon_y^2 = 8$. Since $\varepsilon_x = \varepsilon_y = z$, we have $\varepsilon_f = \sqrt{8} = 2.83$. Therefore $(20 \pm 2) - (19 \pm 2) = 1 \pm 2.83$.

10. Finding extrema.

Find the point (x^*, y^*, z^*) that is at the minimum of the function

$$f(x, y, z) = 2x^2 + 8y^2 + z^2$$

subject to the constraint equation

$$g(x, y, z) = 6x + 4y + 4z - 72 = 0.$$

Use the Lagrange multiplier method:

$$\begin{aligned}
\left(\frac{\partial f}{\partial x}\right) - \lambda\left(\frac{\partial g}{\partial x}\right) &= 0, \\
\left(\frac{\partial f}{\partial y}\right) - \lambda\left(\frac{\partial g}{\partial y}\right) &= 0, \\
\left(\frac{\partial f}{\partial z}\right) - \lambda\left(\frac{\partial g}{\partial z}\right) &= 0.
\end{aligned}$$

This gives

$$\begin{aligned}
(4x) - \lambda(6) &= 0, \\
(16y) - \lambda(4) &= 0, \\
(2z) - \lambda(4) &= 0;
\end{aligned}$$

i.e.,

$$
\begin{aligned}
\lambda &= \frac{2x}{3}, \\
\lambda &= 4y, \\
\lambda &= \frac{z}{2}.
\end{aligned}
$$

$$g(x, y, z) = 6x + 4y + 4z - 72 = 0.$$

Find x:

$$
\begin{aligned}
0 &= 6x + 4y + 4z - 72 \\
&= 6x + 4\left(\frac{2x}{12}\right) + 4\left(\frac{4x}{3}\right) - 72 \\
&= \left(\frac{36x}{3}\right) - 72, \\
x^* &= 6
\end{aligned}
$$

Find y:

$$
\begin{aligned}
0 &= 6x + 4y + 4z - 72 \\
&= 6(6) + 4y + 4(8y) - 72 \\
&= -36 + 36y, \\
y^* &= 1
\end{aligned}
$$

Find z:

$$
\begin{aligned}
0 &= 6x + 4y + 4z - 72 \\
&= 6(6) + 4(1) + 4z - 72 \\
&= -32 + 4z, \\
z^* &= 8.
\end{aligned}
$$

(d) Yes, (b) and (c) should be equal. V is a state function, as shown in (b). Therefore, the path of integration should not affect the final result.

13. Equations of state.

Which of the following could be the total derivative of an equation of state?

(a) $$\frac{2nRT}{(V-nb)^2}\,dV + \frac{R(V-nb)}{nb^2}\,dT.$$

(b) $$-\frac{nRT}{(V-nb)^2}\,dV + \frac{nR}{V-nb}\,dT.$$

Cross-derivatives for equations of state must be equal.

(a) Is $$\frac{\partial}{\partial T}\left[\frac{2nRT}{(V-nb)^2}\right] = \frac{\partial}{\partial V}\left[\frac{R(V-nb)}{nb^2}\right]?$$

$$\frac{2nRT}{(V-nb)^2} \neq \frac{R}{nb^2}.$$

No, this could not be the total derivative of an equation of state.

(b) Is $$\frac{\partial}{\partial T}\left[-\frac{nRT}{(V-nb)^2}\right] = \frac{\partial}{\partial V}\left(\frac{nR}{V-nb}\right)?$$

$$-\frac{nR}{(V-nb)^2} = -\frac{nR}{(V-nb)^2}.$$

Yes, this could be the total derivative of an equation of state.

18. Short-answer questions.

(a) Compute the partial derivatives $(\partial f/\partial x)_y$ and $(\partial f/\partial y)_x$ for the following functions:

(i) $f(x,y) = \ln(2x) + 5y^3$,

(ii) $f(x,y) = (x+a)^8 y^{1/2}$,

(iii) $f(x,y) = e^{7y^2} + 9$,

(iv) $f(x,y) = 13x + 6xy^3$.

(b) Which of the following are exact differentials?

(i) $5x^2\,dx + 6y\,dy$,

(ii) $5\ln(y)\,dx + 5x^0\,dy$,

(iii) $(\sin x - y\cos x)\,dx + \sin(-x)\,dy$,

(iv) $(e^{2x} + y)\,dx + (x + e^{2y})\,dy$,

(v) $y^2\,dy + x^2\,dx$.

(a) (i) $\left(\dfrac{\partial f}{\partial x}\right)_y = \dfrac{1}{x}, \quad \left(\dfrac{\partial f}{\partial y}\right)_x = 15y^2;$

(ii) $\left(\dfrac{\partial f}{\partial x}\right)_y = 8y^{1/2}(x+a)^7, \quad \left(\dfrac{\partial f}{\partial y}\right)_x = \dfrac{(x+a)^8}{2\sqrt{y}};$

(iii) $\left(\dfrac{\partial f}{\partial x}\right)_y = 0, \quad \left(\dfrac{\partial f}{\partial y}\right)_x = 14ye^{7y^2};$

(iv) $\left(\dfrac{\partial f}{\partial x}\right)_y = 13 + 6y^3, \quad \left(\dfrac{\partial f}{\partial y}\right)_x = 18xy^2.$

(b) Let $f(x,y) = u(x,y)\,dx + v(x,y)\,dy$, where $f(x,y) = dg(x,y)$ if $f(x,y)$ is an exact differential.

	$\dfrac{\partial u}{\partial y}$	$\dfrac{\partial v}{\partial y}$	
(i)	0	0	exact
(ii)	$\dfrac{5}{y}$	0	inexact
(iii)	$-\cos x$	$-\cos x$	exact
(iv)	1	1	exact
(v)	0	0	exact