User's Manual for **GPGLP -** A Posynomial Geometric Programming Solver

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1 Introduction

1.1 Overview

GPGLP is a computer code that finds the minimum of a multi-variable, nonlinear function subject to a set of constraints, all expressed in standard posynomial geometric programming form. Details about the algorithm used may be found in Rajgopal [1]. It is coded in FORTRAN and is linked with the XMP linear programming library [2]. This version of GPGLP runs on an IBM-386 or better microcomputer (or compatible), under either MS-DOS or MS-WINDOWS operating systems. For running this software on other platforms, please contact to the author. This manual contains all instructions for installing the software, entering data for a geometric programming model, running the software and analyzing the output.

Posynomial Geometric Programming

Geometric Programming (GP) is a special form of nonlinear programming with applications in a variety of areas; it is a particularly powerful method for engineering design and optimization problems. GP was first introduced by Duffin, Peterson and Zener [3] in their seminal book, published in 1967. This early work is also an excellent reference in this field, and provides the mathematical theory of geometric programming and illustrates its applications to engineering design problems. Another important source of information is the book by Beightler and Phillips [4] which takes a somewhat more applied approach and contains a collection of

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[1] Rajgopal, Jayant "An Algorithm for Solving the Posynomial GP Problem, Based on Generalized Linear Programming," *Technical Report No. TR95-10,* Department of Industrial Engineering, University of Pittsburgh, Pittsburgh, PA.

[2] Marsten, Roy E., "The Design of the XMP Linear Programming Library," *ACM Transactions on Mathematical Software*, Vol. 7, No. 4, December 1981, pp.481-497.

[3] Duffin, R.J., E.L. Peterson and C.M. Zener, *Geometric Programming*, John Wiley & Sons, New York, 1967.

[4] Beightler C. and D.T. Phillips, *Applied Geometric Programming*, John Wiley & Sons, New York, 1976.

various sources of information up to that date along with some additions to the book of Duffin et al.

Numerous applications of geometric programming can be found in the paper by Ecker [5] who gives a complete application oriented survey of geometric programming with an excellent bibliography. Engineering design is the most natural application for GP since in most design problems, the total cost can be expressed as the sum of individual component costs, each of which is in generally some power function of the design variables. This is the format required by GP; the individual components of the additive polynomial cost function are expressed as the product of a cost coefficient and individual design variables, each raised to some constant power. The exponent a_{ij} of variable j in term i can be any arbitrary real constant, while the sign of the coefficient C_i for term i determines whether the polynomials is a *posynomial* (if C_i is positive) or a *signomial* (if C_i is negative). A geometric program also requires each of the constraint functions to be a posynomial or a signomial which is restricted to be either less than or greater than equal to some constant. The software described in this manual deals with a class of GP problems known as posynomial geometric programs: the objective and constraint functions are all posynomials, and each constraints function is restricted to be less than or equal to 1. Although this structure may appear to be restrictive, there are many applications that naturally lead to such an additive formulation, and others which can be stated in this format through simple techniques such as a change of variables or algebraic manipulation - a good reference to such instances is the book by Beightler and Phillips.

A posynomial geometric programming problem having N variables and M constraints can be represented as:

$$\begin{aligned} \min \quad & \sum_{t=1}^{T_0} C_{0t} \prod_{n=1}^{N} x_n^{a_{0tn}} \\ s.t. \quad & \sum_{t=1}^{T_m} C_{mt} \prod_{n=1}^{N} x_n^{a_{mnt}} \quad m = 1,.., M \\ & x_n > 0 \end{aligned}$$

where T_0 : # of terms in the objective

T_m : # of terms in constraint m, $\forall m$

Coefficients C_{0t} , C_{mt}>0 , $\forall m$

Exponents a_{0nt}, a_{mnt} unrestricted in sign, $\forall m$

[5] Ecker, JG., "Geometric Programming: Methods, Computations and Applications", *SIAM Review*, Vol. 22, No. 3, July 1980, pp. 338-362.

1.2 Organization

This users guide is organized into four chapters including this introductory chapter. The modeling and programming framework of GPGLP is described in Chapter 2 by describing the main subroutines and their interactions. Following that, Chapter 3 presents the hardware requirements, a description of the distribution disk contents and stepwise installation guidelines. Chapter 4 describes the GPGLP features related to program execution and gives simple examples for data entry and program execution. Solution output from GPGLP is also explained in Chapter 4. This guide also contains several test problems, all of which listed in Appendix B.

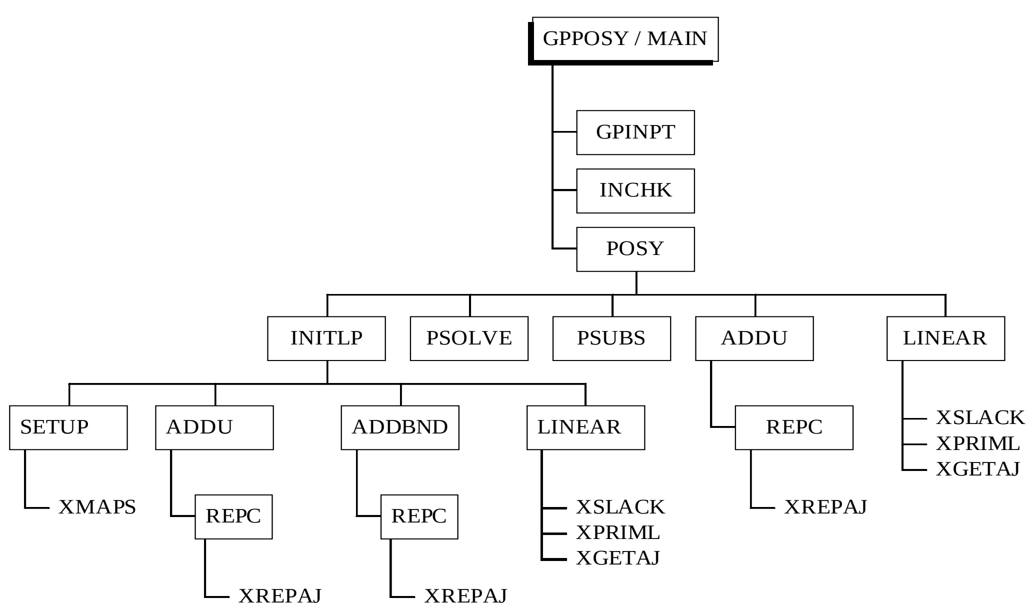
2 Implementation of GPGLP

GPGLP is coded in FORTRAN and is linked with the XMP linear programming library through an interface module. The algorithm used is based upon the generalized linear programming approach used by Dantzig. Details of the algorithm may be found in Rajgopal [1], but the procedure is summarized as follows. First the primal-dual pair in GP are linearized so that the primal is stated as a semi-infinite linear program while the dual is a generalized linear program (GLP). The latter is a linear program where the columns corresponding to the decision variables each come from some convex set. The procedure begins with an initial approximation of the GLP as a linear program with some initial set of columns, which is then solved using the simplex method. At each iteration the dual prices at the current iteration (simplex multipliers) are used to price out columns and attractive columns are added to the LP to obtain a better approximation; in practice, we replace columns that are not in use (as opposed to adding columns) so that the size of the LP approximation does not grow. The LP is then reoptimized and the procedure continues until no columns price out favorably. At this point the procedure stops and the optimal primal solution is recovered from the final dual tableau. It should be noted that the process of pricing columns in the dual is equivalent to looking for feasibility in the primal, so that at each iteration an estimate is available for the current values of the primal objective and constraints and the progress of the algorithm may be monitored. Again, the reader is encouraged to read the descriptive paper by Rajgopal for details on the procedure; a postscript copy of this paper may be obtained via anonymous ftp from ***ftp.pitt.edu*** in the subdirectory ***/dept/ie***, where the file containing this paper resides under the name ***gpglp.pdf***.

Personal computer (MS-DOS®/WINDOWS®) versions of GPGLP do not assume any pre-set bounds on the size of the problem. Memory is allocated dynamically for the required data structures. The size of the problem to be solved is not internally restricted by the program, rather it depends on the total available memory of the particular computer being used. GPGLP uses the extended memory manager (DOSXMSF.EXE) developed by Microsoft® to allocate memory for variables and arrays. This implementation allows for solving large scale problems.

The subroutine organization chart displays the hierarchy of the various subroutines that make up the GPGLP system. All subroutines in boxes are either specific to the algorithm or are used to interface with the XMP routines. The names that are not in boxes correspond to XMP subroutines that are called by the interface. These in turn call other XMP subroutines to do all the work associated with the simplex method, the interested reader is referred to the manual for XMP.

The main program (GPPOSY) reads in data either from a user specified datafile or from the keyboard through a set of interactive data input procedures in subroutine GPINPT. It then computes, checks and allocates the required memory via subroutine INCHK. If the memory specifications of the particular computer do not permit running the problem, GPGLP reports an error and terminates. After successful memory allocation, it then calls the main solution subroutine POSY. The algorithm sets up the initial linear program via subroutine INITLP. To solve this LP and the LP's at each successive iteration, an interface routine SETUP is used at this point to call XMP and initialize the data structures, arrays and the memory requirements. Next, columns are generated from an initial set of dual vectors for each posynomial and added on the



GPGLP Subroutine Organization

LP via subroutine ADDU, and finally columns are also added for any simple upper and lower bounding constraints via subroutine ADDBND. Finally the initial LP is solved by having the interface subroutine LINEAR call the required XMP routine. The main algorithm iterates by adding new column via ADDU and solving the resulting LP via LINEAR. When the algorithm converges, subroutine PSOLVE solves for the primal variables by exponentiating the simplex multipliers corresponding to the final LP approximation of the dual, and another routine PSUBS is called to compute the value of each term of each posynomial. The final solution is then printed out to a file and the procedure halts.

3 Installation

3.1 Hardware Requirements

AT a minimum GPGLP requires an Intel® 80386 based microcomputer running DOS or WINDOWS. It uses the available extended memory so that the size of the problems that it can solve depends only on the total available memory. A 486 or 586 based processor is recommended since the algorithm is computationally intensive. Furthermore, a minimum of 4MB of RAM is recommended since an extended memory manager (DOSXMSF.EXE) is supplied. GPGLP is an MS-DOS® executable program but can also be run through Windows® as a DOS executable. is also available for other computing platforms and the FORTRAN source files can be requested from the author. Disk space requirement is minimal (less than 1MB) but very large problems may require more disk space to store input and output datafiles.

GPGLP is developed in FORTRAN and compiled with Microsoft FORTRAN Visual Workbench v1.0®. It is an MS-DOS executable program and hence does not need a FORTRAN compiler to run it.

3.2 Distribution Disk

The distribution disk contains one self extracting archive file named GPINSTAL.EXE. This archive file is created by PKZIP and ZIP2EXE, both public domain file compression utilities. This archive file contains all of the files required to run GPGLP together with this users manual which is typed in Microsoft Word 6.0® and datafiles for various test problems that are given in Appendix A. This file GPINSTAL.EXE should be copied to the computer's hard disk (preferably into its own subdirectory) and executed by typing *GPINSTAL*. The files will shown below will be extracted automatically into the current subdirectory; at this point GPINSTAL.EXE (the original archive file) may be deleted from to save disk space. GPGLP may also be run from a floppy diskette but this is not recommended since execution speed will be significantly slower.

GPGLP.EXE	The executable file for the posynomial solver. This file is created by Microsoft FORTRAN compiler.
DOSXMSF.EXE	Extended memory manager required to be in the same directory with GPGLP.EXE.
PARMS.POS	The parameter file for GPGLP. If this file exists in the current directory, GPGLP reads in some parameter values, if not, defaults are assumed.
GPREAD.DOC	This users manual (in MS Word 6.0 format)
GPREAD.TXT	This users manual (in text format)
GPREAD.PDF	This users manual in postscript format, can be copied directly to a postscript printer.
*.DAT	Various example problems (input datafiles, text)
*.OUT	Various example problems (output datafiles, text)

GPGLP files contained in the distribution disk archive

4 Program Execution

4.1 Running GPGLP

For running the program, simply type *GPGLP* at the DOS prompt. The program accepts input either from the terminal or from a data input file, both of which are described with examples below. First, we describe the case where the problem parameters are entered through the keyboard. Consider the example geometric programming formulation below.

$$\begin{array}{ll} \min & 10x_1^{-1}x_2 \\ s.t & 3x_1^{0.5} \le 1 \\ & x_1 x_2^{-2} \le 1 \\ & x_i \ge 0 \end{array}$$

We now enter this example problem. The figure on the right presents the dialog used to input data to GPGLP. User responses will be distinguished by characters in italics.

First, the user is prompted for the input format. Here, the user selects the terminal input format (*1*) and GPGLP prompts the user for the required data. The user first provides a data *filename*, if the file name has no extension GPGLP assumes '.DAT'. Next , the problem may be assigned a name via *problem name*; this is however, optional. The program then prompts for the number of variables and constraints. Blanks, commas or carriage returns are acceptable to separate fields. The *number of terms* in each posynomial is entered next. Following this coefficient and exponent data is entered term by term starting with the first term in the objective and ending with the last term in the last constraint. For each term, its *coefficient*, the *number of variables that have nonzero exponents in that term*, and the values of the *exponents* for each of these variables are entered. Finally, simple upper and lower bounds on variables are entered. A value of *0* indicates no bounds of the specified kind. If bounds are present, the number of variables that have bounds is entered followed by the variable index and the corresponding bound. This is done for both and upper and lower bounds as required. Finally the program prompts the user for the maximum number of iterations and the frequency (in iterations) with which intermediate reports are desired.

Creating a new model

```
****************************************
* GEOMETRIC PROGRAMMING *
*         VIA GENERALIZED         *
*     LINEAR PROGRAMMING     *
****************************************
Input via terminal (=1) or file (=2) ? 1
Enter filename for storing data : TEST
Enter the problem name  :
             An example problem
Enter NVAR (# of variables)
       MCON (# of constraints)  :  2  2
Enter # of terms in each posynomial
             Posynomial No. 0 :  1
             Posynomial No. 1 :  1
             Posynomial No. 2 :  1

 ****COEFFICIENTS AND EXPONENTS****
FIRST Enter the term coefficient c(i)
THEN number of variables that appear in that term
THEN nonzero exponents for all variables in term i
    Enter these as the pair j,a(i,j) where
                    j=variable index
               a(i,j)=Exponent value
%%% In Posynomial no. 0
Coefficient for term 1       : 10
No. of variables in term 1 :  2
NONZERO EXPONENTS
   : 1  -1
   : 2   1
%%% In Posynomial no. 1
Coefficient for term 1       :  3
No. of variables in term 1 :  1
NONZERO EXPONENTS
   : 1  0.5
 %%% In Posynomial no. 2
Coefficient for term 1       :  1
No. of variables in term 1 :  2
NONZERO EXPONENTS
   : 1   1
   : 2  -2

No. of variables with lower bounds :  0
No. of variables with upper bounds :  0

NOTE: Input data has been written on I/O UNIT 7 and
saved under the file name TEST.DAT
Enter limit on no. of iterations to be performed &
frequency (in iterations) with which reports are required
     : 100 100
OUTPUT DATA FILE   :  TEST.OUT
Stop - Program terminated
```

After the data input is complete, GPGLP saves the input data into a text file (in this example: ‘TEST.DAT’). It then solves the model and generates an output file called ‘TEST.OUT’. This output file is a text file that contains the problem formulation together with the best solution found, and may be edited using any text editor and/or copied to a printer to view the results.

One can also run a previously entered and saved problem. As an example for running a previously entered problem, consider the existing datafile P01.DAT. The default input file extension is ‘.DAT’. The datafile contains the same information entered in the same order as in the interactive mode. The data may be stored in the file in free format. For this example, a value of *100* is entered for both the iteration limit and for the reporting frequency. GPGLP then solves the model and generates an output file called ‘P01.OUT’. Again, this output file is a text file that contains the problem formulation together with the best solution found.

```
Solving an existing model: (eg. P01.DAT)

***************************************
* GEOMETRIC PROGRAMMING *
*        VIA GENERALIZED        *
*     LINEAR PROGRAMMING    *
***************************************
Input via terminal (=1) or file (=2) ? 2
Enter filename for storing data    : P01
Enter limit on no. of iterations to be
performed & frequency (in iterations) with
which reports are required
                : 100 100
 OUTPUT DATA FILE   :  P01.OUT
 Stop - Program terminated
```

4.3 GPGLP Parameters

GPGLP has only two user specified parameters that are required for program execution: the limit on the number of iterations to be performed and the frequency with which the reports are required. (Recall that for the previous two examples we have entered 100 for both of these parameters meaning that GPGLP will run a maximum of 100 iterations and report the solution only once after 100 iterations are done.) The first parameter allows the user has the ability to stop the algorithm after a few iterations or let it run as long as required. Larger problem will need more iterations than smaller one, but in general there is no harm in entering some large number (such as 5000 or 10,000) for the maximum limit - the algorithm will terminate either when an optimum solution that meets the convergence criteria is found, or when no further improvement is possible with the given convergence criteria. The second parameter allows the user to track the progress being made by the algorithm towards the optimum solution, if only the final solution is desired the same values may be entered for both parameters.

There is another set of parameters which may be optionally specified. The default values for these are contained in the text file ‘PARMS.POS’ which is required for execution. There are 5 parameters in this file. These are:

IPRINT	: 1 if debugging output is required, else 0
IPRG	: 1 if RHO values are to be printed as entered, else 0
IPRX	: 1 if primal solution estimate is required at each iteration, else 0
ALLOW	: Maximum allowable violation for a primal constraint
RGAP	: Convergence criterion

GPGLP Parameters

The parameters in this file are either for debugging and informational purposes (*IPRINT, IPRG, IPRX*), or for specifying the required solution precision (*ALLOW, RGAP*). These values can be changed by simply editing this file with a text editor and changing the parameter values. This file is **formatted**, values must be typed after the 50th column. If this file is missing in the current directory, GPGLP assumes default values and executes accordingly. The default values for the debugging parameters are all 0; and for the precision variables, *ALLOW* is set to 0.0000001 and *RGAP* is set to 0.000001.

Debugging parameters (*IPRINT, IPRG, IPRX*) are especially helpful in examining the solution progress. If *IPRINT* is 1 then a vast amount of debugging output is printed out; much of this comes from XMP. *IPRG* and *IPRX* are used to print out the values of the dual variables hat generate the columns at each iteration and details on the primal solution. In general, there is no benefit to the casual user in specifying a value other than default of 0 for these parameters. The parameter values for solution precision may also be changed. *RGAP* refers to the maximum acceptable value for the ratio (*primal objective* - *dual objective)/(dual objective)* and is a measure of convergence, while *ALLOW* is the maximum acceptable violation by each constraint. Larger values may lead to premature convergence whereas values that are too small may result in excessive computing times for little additional benefit. In certain cases with very small values for these two parameters, if no further improvement is achievable within the precision of the computer, the program stops and prints out the best solution found. Thus once again, there is no harm in specifying an extremely small value such as 10^{-10} for example, and the program will halt after the doing its best.

4.4 GPGLP Output

The GPGLP output file name is the same as the input file name except for the extension which is '.OUT'. This output file contains the problem formulation, any intermediate solutions requested and the final solution. The first portion of this output file contains problem data summary, followed by the initial solution. The second part of the output contains information about the optimum. The list of active columns is printed out (in case there is an interest in these, they may be further examined by using *IPRG*=1; for the layperson they are of not much use), followed by the best solution found by GPGLP. The optimal values of the primal variable and the values of all posynomial as well as of each term contained in them are reported. The output also contains the value of the primal objective at the optimum solution and the value of the final LP approximation to the GP dual (which is the same as the dual objective). For an example see the ‘P01.OUT’ file on the next page. Note that although an iteration limit of 100 was specified, the program stopped after 22 iterations since the optimum for the given tolerances is achieved.

```
RUN DATE (M/D/Y) :  2/27/1995

 SUMMARY OF PROBLEM DATA
    PROBLEM NAME: P01.DAT
    NUMBER OF VARIABLES      3
    NUMBER OF CONSTRAINTS    1
    TOTAL NUMBER OF TERMS    9
    DEGREES OF DIFFICULTY    5

 POSY   TERM   COEFFICIENT          EXPONENTS
    0       1         .50000D+01       1.0000   .0000   .0000
    0       2         .50000D+05      -1.0000   .0000   .0000
    0       3         .20000D+02        .0000  1.0000   .0000
    0       4         .72000D+05        .0000 -1.0000   .0000
    0       5         .10000D+02        .0000   .0000  1.0000
    0       6         .14400D+06        .0000   .0000 -1.0000
    1       7         .40000D+01      -1.0000   .0000   .0000
    1       8         .32000D+02        .0000 -1.0000   .0000
    1       9         .12000D+03        .0000   .0000 -1.0000

**************************************************
INITIAL LP SOLUTION IS       7.090077 =LOG(    .120000E+04)
 ****OPTIMUM SOLUTION****
ACTIVE GLP COLUMNS (POSY<0 IMPLIES A BOUNDING CONSTRAINT):
   ID POSY    LAMBDA
    2   1     .36177742E+00
    3   0     .60750412E+00
    4   0     .33668371E+00
    6   0     .55812171E-01

          ********************
          * REPORT OF OPTIMAL *
          * PRIMAL VARIABLES  *
          ********************
PRIMAL VARIABLES ARE
              I            X(I)
              1            108.85942
              2            85.057310
              3            204.41592

 POSYNOMIAL NUMBER    0     VALUE=     6299.8447
 VALUES OF TERMS ARE
    544.29711       459.30797        1701.1462         846.48809         2044.1592
    704.44610
 POSYNOMIAL NUMBER    1     VALUE=     .99999998
 VALUES OF TERMS ARE
    .36744638E-01     .37621693        .58703842

 PRIMAL SOLUTION     .62998447E+04
 LP SOLUTION     .62998397E+04
 CONVERGENCE CRITERION     .78877016E-06

 TOTAL NUMBER OF ITERATIONS =    22
 TOTAL NUMBER OF COLUMNS GENERATED=   53

 CPU TIME =    .0500 SECONDS
```

Problem Data

Initial Solution

Best Solution Found

A Sample Test Problem Output (P01.OUT)

APPENDIX A: TEST PROBLEMS

This appendix lists a number of GP problems on which GPGLP has been successfully tested. These problems come from several different sources which are listed below. For each problem the minimum value found by GPGLP for the objective function is also listed.

Problem P1: Smith, S.B., "Economic Lot Sizing with a Restriction on Setup Hours," *Production and Inventory Management*, Vol. 11, pp. 82-88, 1970.

Problem P2: Beightler, C.S. and D.T. Phillips , *Applied Geometric Programming*, John Wiley and Sons, NY, 1976.

Problem P3: Source unknown.

Problem P4: Zangwill, W.I., "The Convex Simplex Method," *Management Science*, Vol. 14, pp. 221-238, 1967.

Problem P5: Kundapur, S., "Minimum Weight Design of Reinforced Concrete Floor System Using Geometric Programming," *Technical Report,* Department of Industrial and Management Engineering, University of Iowa, Iowa City, 1976.

Problem P6: Passy, U. and D.J.Wilde, "A Geometric Programming Algorithm for Solving Chemical Equilibrium Problems," *SIAM Journal of Applied Mathematics*, Vol. 16, pp. 363-373, 1968.

Problem P7: Dembo, R.S., "A Set of Geometric Programming Test Problems and Their Solutions," *Mathematical Programming*, Vol. 10, pp. 192-213, 1976.

Problem P8: *ibid.*

Problem P9: *ibid.*

Problem P10: Rijckaert, M.J. and X.M.Martens, "Comparison of Generalized Geometric Programming Algorithms," *Journal of Optimization Theory and Applications*, Vol. 26, No. 2, 1978.

Problem P10A: *ibid.*

Problem P11: *ibid.*

Problem P12: Dembo, R.S., "A Set of Geometric Programming Test Problems and Their Solutions," *Mathematical Programming*, Vol. 10, pp. 192-213, 1976.

Problem P13: Rijckaert, M.J. and X.M.Martens, “Comparison of Generalized Geometric Programming Algorithms,” *Journal of Optimization Theory and Applications*, Vol. 26, No. 2, 1978.

Problem P14: Jha, S., Kortanek,K.O. and H.No., “Lotsizing and Setup Time Reduction under Stochastic Demand: A Geometric Programming Approach,” *Working Paper Series No. 88-12*, Department of Management Science, University of Iowa, IA.

Problem P15: Fiacco, A.V. and A.Ghaemi, “Sensitivity Analysis of a Nonlinear Water Pollution Control Model Using an Upper Hudson River Data Base,” *Operations Research*, Vol. 30(1), 1982.

Problem P16: Fiacco, A.V. and A.Ghaemi, “Sensitivity and Parametric Bound Analysis of an Electric Power Generation GP Model: Optimal Steam Turbine Exhaust Annulus and Condenser Sizes,” *Working Paper T-437*, School of Engineering and Applied Science, George Washington University, Washington, D.C.

Posynomial geometric programming test problems

N: Number of variables / M: Number of constraints / D: Degrees of difficulty

Problem P1 N=3 M=1 D=5

$$\min \quad 5x_1 + 50000x_1^{-1} + 20x_2 + 72000x_2^{-1} + 10x_3 + 144000x_3^{-1}$$

$$s.t.$$

$$4x_1^{-1} + 32x_2^{-1} + 120x_3^{-1} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,3$$

Optimum solution = 6299.8434

Problem P2 N=3 M=2 D=1

$$\min \quad x_1^{-1}x_2^{-1}x_3 + 25x_1 x_2^{-1}$$

$$s.t.$$

$$x_1 x_2 \le 1$$

$$5x_1 x_3^{-1} + x_2 x_3^{-1} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,3$$

Optimum solution = 10.700981

Problem P3 N=4 M=2 D=1

$$\min \quad 0.49x_1 x_2^{0.6667} x_3^3 + 200x_1 x_2 + 0.73x_1 x_2^{0.6667} x_3^3 x_4$$

$$s.t.$$

$$0.2740 * 10^7 x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1} \le 1$$

$$0.6667 * 10^{-6} x_2 x_3 x_4 + x_4 \le 1$$

$$x_i \ge 0 \qquad i = 1,..,4$$

Optimum solution = 71523512

Problem P4 N=3 M=3 D=6

min $x_1^{-1}x_2^{-1}x_3^{-1}$

$s.t.$

$$2x_1 + x_2 + 3x_3 \le 1$$
$$x_1 + 3x_2 + 2x_3 \le 1$$
$$x_1 + x_2 + x_3 \le 1$$

$$x_i \ge 0 \qquad i = 1,..,3$$

Optimum solution = 202.77746

Problem P5 N=5 M=8 D=13

min $(81.833W)x_1 + (216.25W)x_2 + (216.25W)x_3 + (8.167W)x_4 + (1.402W)x_5$

$s.t.$

$$0.87x_1^{-1} + 10.11x_1^{-2} \le 1$$
$$1.58x_1^{-1} \le 1$$
$$8.728x_1^{-2} + 71.273x_1^{-3} \le 1$$
$$0.008x_1 x_2^{-1} \le 1$$
$$0.0018x_1 x_3^{-1} \le 1$$
$$(2.7065*10^{-7}*W^2)x_4^{-1} + (4.6083*10^{-4}*W^2)x_1 x_4^{-2} + (5.8371*10^{-3}*W^2)x_4^{-2} \le 1$$
$$(0.9877*10^{-8}*W^3)x_4^{-2} + (1.5687*10^{-5}*W^3)x_1 x_4^{-3} + (1.7940*10^{-4}*W^3)x_4^{-3} \le 1$$
$$0.112x_4 x_5^{-1} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,6 \qquad\qquad W = 240$$

Optimum solution cost = 140737.23

Problem P6 N=3 M=1 D=7

min $(10^{16})x_1^{-2}x_2^{-1}x_3^{-1}$

$s.t.$

$$0.4411x_1 + 28.46x_1^2 + 616x_1^2x_2 + 0.03703x_3 + 710.7x_3^2$$
$$+ 0.3225x_1x_3 + 2.93x_2x_3 + 0.04471x_2 + 0.3796x_2^2 + 4.289x_1x_2 \le 1$$

$$x_i \ge 0 \qquad i = 1,..,3$$

Optimum solution = $6.2899*10^{20}$

Problem P7 N=7 M=4 D=10

$$\min \quad 10x_1 x_2^{-1}x_4^2 x_6^{-3}x_7^{-0.25} + 15x_1^{-1}x_2^{-2}x_3 x_4 x_5^{-1}x_7^{-0.5} + 20x_1^{-2}x_2 x_4^{-1}x_5^{-2}x_6 + 25x_1^2 x_2^2 x_3^{-1}x_5^{0.5}x_6^{-2}x_7$$

$s.t.$

$$0.5x_1^{0.5}x_3^{-1}x_6^{-2}x_7 + 0.7x_1^3 x_2 x_3^{-2}x_6 x_7^{0.5} + 0.2x_2^{-1}x_3 x_4^{-0.5}x_6^{0.6667}x_7^{0.25} \le 1$$

$$1.3x_1^{-0.5}x_2 x_3^{-1}x_5^{-1}x_6 + 0.8x_3 x_4^{-1}x_5^{-1}x_6^2 + 3.1x_1^{-1}x_2^{0.5}x_4^{-2}x_5^{-1}x_6^{0.3333} \le 1$$

$$2x_1 x_3^{-1.5}x_5 x_6^{-1}x_7^{0.3333} + 0.1x_2 x_3^{-0.5}x_5 x_6^{-1}x_7^{-0.5} + x_1^{-1}x_2 x_3^{0.5}x_5 + 0.65x_2^{-2}x_3 x_5 x_6^{-1}x_7 \le 1$$

$$0.2x_1^{-2}x_2 x_4^{-1}x_5^{0.5}x_7^{0.3333} + 0.3x_1^{0.5}x_2^2 x_3 x_4^{0.3333}x_5^{-0.6667}x_7^{0.25} + 0.4x_1^{-3}x_2^{-2}x_3 x_5 x_7^{0.75} + 0.5x_3^{-2}x_4 x_7^{0.5} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,7$$

Optimum solution = 1809.98

Problem P8 N=7 M=4 D=10

$$\min \quad 10x_1^1 x_2^{-1}x_4^2 x_6^{-3}x_7^{0.125} + 15x_1^{-1}x_2^{-2}x_3 x_4 x_5^{-1}x_7^{-0.5} + 20x_1^{-2}x_2 x_4^{-1}x_5^{-2}x_6 + 25x_1^2 x_2^2 x_3^{-1}x_5^{0.5}x_6^{-2}x_7$$

$s.t.$

$$0.5x_1^{0.5}x_3^{-1}x_6^{-2}x_7 + 0.7x_1^3 x_2 x_3^{-2}x_6 x_7^{0.5} + 0.2x_2^{-1}x_3 x_4^{-0.5}x_6^{0.6667}x_7^{0.25} \le 1$$

$$1.3x_1^{-0.5}x_2 x_3^{-1}x_5^{-1}x_6 + 0.8x_3 x_4^{-1}x_5^{-1}x_6^2 + 3.1x_1^{-1}x_2^{0.5}x_4^{-2}x_5^{-1}x_6^{0.3333} \le 1$$

$$2x_1 x_3^{-1.5}x_5 x_6^{-1}x_7^{0.3333} + 0.1x_2 x_3^{-0.5}x_5 x_6^{-1}x_7^{-0.5} + x_1^{-1}x_2 x_3^{0.5}x_5 + 0.65x_2^{-2}x_3 x_5 x_6^{-1}x_7 \le 1$$

$$0.2x_1^{-2}x_2 x_4^{-1}x_5^{0.5}x_7^{0.3333} + 0.3x_1^{0.5}x_2^2 x_3 x_4^{0.3333}x_5^{-0.6667}x_7^{0.25} + 0.4x_1^{-3}x_2^{-2}x_3 x_5 x_7^{0.75} + 0.5x_3^{-2}x_4 x_7^{0.5} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,7$$

Optimum solution = 911.9900

Problem P9 N=7 M=4 D=10

$$\min \quad 10x_1^1 x_2^{-1}x_4^2 x_6^{-3}x_7^{0.5} + 15x_1^{-1}x_2^{-2}x_3 x_4 x_5^{-1}x_7^{-0.5} + 20x_1^{-2}x_2 x_4^{-1}x_5^{-2}x_6 + 25x_1^2 x_2^2 x_3^{-1}x_5^{0.5}x_6^{-2}x_7$$

$s.t.$

$$0.5x_1^{0.5}x_3^{-1}x_6^{-2}x_7 + 0.7x_1^3 x_2 x_3^{-2}x_6 x_7^{0.5} + 0.2x_2^{-1}x_3 x_4^{-0.5}x_6^{0.6667}x_7^{0.25} \le 1$$

$$1.3x_1^{-0.5}x_2 x_3^{-1}x_5^{-1}x_6 + 0.8x_3 x_4^{-1}x_5^{-1}x_6^2 + 3.1x_1^{-1}x_2^{0.5}x_4^{-2}x_5^{-1}x_6^{0.3333} \le 1$$

$$2x_1 x_3^{-1.5}x_5 x_6^{-1}x_7^{0.3333} + 0.1x_2 x_3^{-0.5}x_5 x_6^{-1}x_7^{-0.5} + x_1^{-1}x_2 x_3^{0.5}x_5 + 0.65x_2^{-2}x_3 x_5 x_6^{-1}x_7 \le 1$$

$$0.2x_1^{-2}x_2 x_4^{-1}x_5^{0.5}x_7^{0.3333} + 0.3x_1^{0.5}x_2^2 x_3 x_4^{0.3333}x_5^{-0.6667}x_7^{0.25} + 0.4x_1^{-3}x_2^{-2}x_3 x_5 x_7^{0.75} + 0.5x_3^{-2}x_4 x_7^{0.5} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,7$$

Optimum solution = 543.7360

Problem P10 N=8 M=7 D=3

$$\min \quad 2x_1^{0.9}x_2^{-1.5}x_3^{-3} + 5x_4^{-0.3}x_5^{2.6} + 4.7x_6^{-1.8}x_7^{-0.5}x_8$$

$s.t.$

$$7.2x_1^{-3.8}x_2^{2.2}x_3^{4.3} + 0.5x_4^{-0.7}x_5^{-1.6} + 0.2x_6^{4.3}x_7^{-1.9}x_8^{8.5} \le 1$$
$$10x_1^{2.3}x_2^{1.7}x_3^{4.5} \le 1$$
$$0.2x_4^{-2.1}x_5^{0.4} \le 1$$
$$6.2x_6^{4.5}x_7^{-2.7}x_8^{-0.6} \le 1$$
$$3.1x_1^{1.6}x_2^{0.4}x_3^{-3.8} \le 1$$
$$3.7x_4^{5.4}x_5^{1.3} \le 1$$
$$0.3x_6^{-1.1}x_7^{7.3}x_8^{-5.6} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,8$$

Optimum solution = 29.226450

Problem P10A N=8 M=7 D=3

$$\min \quad 2x_1^{0.9}x_2^{-1.5}x_3^{-3} + 5x_4^{-0.3}x_5^{2.6} + 4.7x_6^{-1.8}x_7^{-0.5}x_8$$

$s.t.$

$$7.2x_1^{-3.8}x_2^{2.2}x_3^{4.3} + 0.5x_4^{-0.7}x_5^{-1.6} + 0.2x_6^{4.3}x_7^{-1.9}x_8^{8.5} \le 1$$
$$10x_1^{2.3}x_2^{1.7}x_3^{4.5} \le 1$$
$$0.6x_4^{-2.1}x_5^{0.4} \le 1$$
$$6.2x_6^{4.5}x_7^{-2.7}x_8^{-0.6} \le 1$$
$$3.1x_1^{1.6}x_2^{0.4}x_3^{-3.8} \le 1$$
$$3.7x_4^{5.4}x_5^{1.3} \le 1$$
$$0.3x_6^{-1.1}x_7^{7.3}x_8^{-5.6} \le 1$$

$$x_i \ge 0 \qquad i = 1,..,8$$

Optimum solution = 29.229483

Problem P11 N=7 M=7 D=40

$$
\begin{aligned}
\min \quad & 10x_1 x_2^{-1}x_3^{-1}x_4 + 20x_1^{-1}x_4^{-1}x_5 x_6 + 30x_2 x_3 x_4 + 100x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}x_6^{-1}x_7^{-1} + 5x_1^2 x_2^2 x_3 x_5 x_6^{1.5} x_7^2 \\
& + 50x_3^{-0.5}x_4^{-0.5}x_5^{-0.5} + 25x_3^2 x_4^2 x_5^{-1}x_6^{-1}x_7^{-1} + 10x_3^{0.5}x_4^{0.5}x_5 x_6 x_7 \\
s.t. \quad & \\
& 0.1x_1^2 x_2^2 x_3 + 0.05x_4 x_5^{0.5} + 0.15x_6^{0.5}x_7^{0.5} \le 1 \\
& 0.1x_1 x_4 x_7 + 0.05x_1 x_2^{-1}x_3^{-1}x_5 x_6 x_7^{0.5} + 0.05x_2^2 x_3^2 x_4^{-1} + 0.15x_1^{-0.5}x_2^{-0.3}x_3 x_5^{0.5} + 0.1x_5 x_6 + 0.1x_4^2 + 0.2x_1 x_2 x_3 \le 1 \\
& 0.1x_1 + 0.1x_2 + 0.1x_3 + 0.1x_4 + 0.1x_5 + 0.1x_6 + 0.1x_7 \le 1 \\
& 0.02x_1^2 + 0.02x_1 x_2 + 0.02x_1 x_3 + 0.02x_1 x_4 + 0.02x_1 x_5 + 0.02x_1 x_6 + 0.02x_1 x_7 \le 1 \\
& 0.01x_1 x_2^{-1} + 0.01x_2 x_3^{-1} + 0.01x_3 x_4^{-1} + 0.01x_4 x_5^{-1} + 0.01x_5 x_6^{-1} + 0.01x_6 x_7^{-1} \le 1 \\
& 0.1x_1 x_3^{-2} + 0.1x_2 x_4^{-2} + 0.1x_3 x_5^{-2} + 0.1x_4 x_6^{-2} + 0.1x_5 x_7^{-2} \le 1 \\
& 0.02x_1^{-0.5}x_3 + 0.02x_2^{-0.5}x_4 + 0.02x_3^{-0.5}x_5 + 0.02x_4^{-0.5}x_6 + 0.02x_5^{-0.5}x_7 \le 1 \\
& \\
& x_i \ge 0 \qquad i = 1,..,7
\end{aligned}
$$

Optimum solution =178.47794

Problem P12 N=12 M=3 D=18

$$
\begin{aligned}
\min \quad & 100000x_1^{-0.0013}x_2^{-0.0023}x_3^{-0.0025}x_4^{-4.67}x_5^{-4.672}x_6^{-0.0081}x_7^{-0.0081}x_8^{-0.005}x_9^{-0.0009}x_{10}^{-0.0009}x_{11}^{-0.0012} \\
s.t. \quad & \\
& 0.05367x_1 + 0.02186x_2 + 0.09773x_3 + 0.006694x_4 x_5 \le 1 \\
& 10^{-6}x_1 + 10^{-5}x_2 + 10^{-6}x_3 + 10^{-10}x_4 + 10^{-8}x_5 x_{12}^{-1} + 10^{-2}x_6 x_{12}^{-1} + 10^{-4}x_7 x_{12} \\
& + 0.109x_4 x_5 + 1.611*10^{-4}x_2 x_5 x_{12}^{-1} + 10^{-23}x_2 x_4 x_5 + 1.93*10^{-6}x_2 x_4^{-1}x_5 x_{12}^{-2} + 10^{-3}x_{10}x_{12}^{-1} \le 1 \\
& 10^{-6}x_1 + 10^{-5}x_2 + 10^{-6}x_3 + 10^{-9}x_4 + 10^{-9}x_5 x_{12}^{-1} + 10^{-3}x_6 + 10^{-3}x_8 + 0.109x_4 x_5 \\
& + 1.611*10^{-5}x_2 x_5 + 10^{-23}x_2 x_4 x_5 + 1.93*10^{-8}x_2 x_4^{-1}x_5 + 10^{-5}x_9 + 1.118*10^{-4}x_1 x_9 + 10^{-4}x_{11} \le 1 \\
& \\
& x_i \ge 0 \qquad i = 1,..,12
\end{aligned}
$$

Optimum solution = 3.1697036

Problem P13 N=4 M=1 D=7

min $592x_1^{0.65} + 582x_1^{0.39} + 1200x_1^{0.52} + 370x_1^{0.22}x_2^{-0.22} + 250x_1^{0.4}x_3^{-0.4} + 210x_1^{0.62}x_3^{-0.62} + 250x_1^{0.4}x_4^{-0.4}$
$+ 200x_1^{0.85}x_4^{-0.85}$

$s.t.$

$$500x_1^{-1} + 50x_2x_1^{-1} + 50x_3x_1^{-1} + 50x_4x_1^{-1} \leq 1$$

$$x_i \geq 0 \qquad i = 1,2,3,4$$

Optimum solution =126303.19

Problem P14 N=30 M=10 D=274

$$
\begin{aligned}
\min\quad & 171.98S_1^{-0.46}+171.98S_2^{-0.46}+237.07S_3^{-0.46}+237.07S_4^{-0.46}+171.98S_5^{-0.46}+237.07S_6^{-0.46}+237.07S_7^{-0.46}\\
& +237.07S_8^{-0.46}+171.98S_9^{-0.46}+171.98S_{10}^{-0.46}+8090.1S_1+9290.1S_2+5845.1S_3+5845.1S_4\\
& +12890.1S_5+7345.1S_6+7345.1S_7+7645.1S_8+17090.1S_9+17690.1S_{10}\\
& +32360.5Q_A^{-1}S_1^2U_1^{-1}+64720.9Q_B^{-1}S_1^2U_1^{-1}+97081.4Q_C^{-1}S_1^2U_1^{-1}+129441.9Q_D^{-1}S_1^2U_1^{-1}+161802.3Q_E^{-1}S_1^2U_1^{-1}\\
& +32360.5Q_F^{-1}S_1^2U_1^{-1}+64720.9Q_G^{-1}S_1^2U_1^{-1}+97081.4Q_H^{-1}S_1^2U_1^{-1}+129441.9Q_I^{-1}S_1^2U_1^{-1}+161802.3Q_J^{-1}S_1^2U_1^{-1}\\
& +37160.5Q_A^{-1}S_2^2U_2^{-1}+74320.9Q_B^{-1}S_2^2U_2^{-1}+111481.4Q_C^{-1}S_2^2U_2^{-1}+148641.9Q_D^{-1}S_2^2U_2^{-1}+185802.3Q_E^{-1}S_2^2U_2^{-1}\\
& +37160.5Q_F^{-1}S_2^2U_2^{-1}+74320.9Q_G^{-1}S_2^2U_2^{-1}+111481.4Q_H^{-1}S_2^2U_2^{-1}+148641.9Q_I^{-1}S_2^2U_2^{-1}+185802.3Q_J^{-1}S_2^2U_2^{-1}\\
& +23380.2Q_F^{-1}S_3^2U_3^{-1}+46760.5Q_G^{-1}S_3^2U_3^{-1}+70140.7Q_H^{-1}S_3^2U_3^{-1}+93520.9Q_I^{-1}S_3^2U_3^{-1}+116901.2Q_J^{-1}S_3^2U_3^{-1}\\
& +23380.2Q_A^{-1}S_4^2U_4^{-1}+46760.5Q_B^{-1}S_4^2U_4^{-1}+70140.7Q_C^{-1}S_4^2U_4^{-1}+93520.9Q_D^{-1}S_4^2U_4^{-1}+116901.2Q_E^{-1}S_4^2U_4^{-1}\\
& +51560.5Q_A^{-1}S_5^2U_5^{-1}+103120.9Q_B^{-1}S_5^2U_5^{-1}+154681.4Q_C^{-1}S_5^2U_5^{-1}+206241.9Q_D^{-1}S_5^2U_5^{-1}+257802.3Q_E^{-1}S_5^2U_5^{-1}\\
& +51560.5Q_F^{-1}S_5^2U_5^{-1}+103120.9Q_G^{-1}S_5^2U_5^{-1}+154681.4Q_H^{-1}S_5^2U_5^{-1}+206241.9Q_I^{-1}S_5^2U_5^{-1}+257802.3Q_J^{-1}S_5^2U_5^{-1}\\
& +29380.2Q_F^{-1}S_6^2U_6^{-1}+58760.5Q_G^{-1}S_6^2U_6^{-1}+88140.7Q_H^{-1}S_6^2U_6^{-1}+117520.9Q_I^{-1}S_6^2U_6^{-1}+146901.2Q_J^{-1}S_6^2U_6^{-1}\\
& +29380.2Q_A^{-1}S_7^2U_7^{-1}+58760.5Q_B^{-1}S_7^2U_7^{-1}+88140.7Q_C^{-1}S_7^2U_7^{-1}+117520.9Q_D^{-1}S_7^2U_7^{-1}+146901.2Q_E^{-1}S_7^2U_7^{-1}\\
& +30580.2Q_F^{-1}S_8^2U_8^{-1}+61160.5Q_G^{-1}S_8^2U_8^{-1}+91740.7Q_H^{-1}S_8^2U_8^{-1}+122320.9Q_I^{-1}S_8^2U_8^{-1}+152901.2Q_J^{-1}S_8^2U_8^{-1}\\
& +68360.5Q_A^{-1}S_9^2U_9^{-1}+136720.9Q_B^{-1}S_9^2U_9^{-1}+205081.4Q_C^{-1}S_9^2U_9^{-1}+273441.9Q_D^{-1}S_9^2U_9^{-1}+341802.3Q_E^{-1}S_9^2U_9^{-1}\\
& +68360.5Q_F^{-1}S_9^2U_9^{-1}+136720.9Q_G^{-1}S_9^2U_9^{-1}+205081.4Q_H^{-1}S_9^2U_9^{-1}+273441.9Q_I^{-1}S_9^2U_9^{-1}+341802.3Q_J^{-1}S_9^2U_9^{-1}\\
& +70760.5Q_A^{-1}S_{10}^2U_{10}^{-1}+141520.9Q_B^{-1}S_{10}^2U_{10}^{-1}+212281.4Q_C^{-1}S_{10}^2U_{10}^{-1}+283041.9Q_D^{-1}S_{10}^2U_{10}^{-1}+353802.3Q_E^{-1}S_{10}^2U_{10}^{-1}\\
& +70760.5Q_F^{-1}S_{10}^2U_{10}^{-1}+141520.9Q_G^{-1}S_{10}^2U_{10}^{-1}+212281.4Q_H^{-1}S_{10}^2U_{10}^{-1}+283041.9Q_I^{-1}S_{10}^2U_{10}^{-1}+353802.3Q_J^{-1}S_{10}^2U_{10}^{-1}\\
& +0.291Q_AU_1^{-1}+0.582Q_BU_1^{-1}+0.874Q_CU_1^{-1}+1.16Q_DU_1^{-1}+1.46Q_EU_1^{-1}\\
& +0.291Q_FU_1^{-1}+0.582Q_GU_1^{-1}+0.874Q_HU_1^{-1}+1.16Q_IU_1^{-1}+1.46Q_JU_1^{-1}\\
& +0.232Q_AU_2^{-1}+0.465Q_BU_2^{-1}+0.697Q_CU_2^{-1}+0.929Q_DU_2^{-1}+1.16Q_EU_2^{-1}\\
& +0.232Q_FU_2^{-1}+0.465Q_GU_2^{-1}+0.697Q_HU_2^{-1}+0.929Q_IU_2^{-1}+1.16Q_JU_2^{-1}\\
& +0.842Q_FU_3^{-1}+1.68Q_GU_3^{-1}+2.53Q_HU_3^{-1}+3.37Q_IU_3^{-1}+4.21Q_JU_3^{-1}\\
& +0.585Q_AU_4^{-1}+1.17Q_BU_4^{-1}+1.75Q_CU_4^{-1}+2.34Q_DU_4^{-1}+2.92Q_EU_4^{-1}\\
& +0.464Q_AU_5^{-1}+0.928Q_BU_5^{-1}+1.39Q_CU_5^{-1}+1.86Q_DU_5^{-1}+2.32Q_EU_5^{-1}\\
& +0.464Q_FU_5^{-1}+0.928Q_GU_5^{-1}+1.39Q_HU_5^{-1}+1.86Q_IU_5^{-1}+2.32Q_JU_5^{-1}\\
& +1.06Q_FU_6^{-1}+2.12Q_GU_6^{-1}+3.17Q_HU_6^{-1}+4.23Q_IU_6^{-1}+5.29Q_JU_6^{-1}\\
& +0.735Q_AU_7^{-1}+1.47Q_BU_7^{-1}+2.20Q_CU_7^{-1}+2.94Q_DU_7^{-1}+3.67Q_EU_7^{-1}\\
& +1.10Q_FU_8^{-1}+2.20Q_GU_8^{-1}+3.30Q_HU_8^{-1}+4.40Q_IU_8^{-1}+5.50Q_JU_8^{-1}\\
& +0.427Q_AU_9^{-1}+0.855Q_BU_9^{-1}+1.28Q_CU_9^{-1}+1.71Q_DU_9^{-1}+2.14Q_EU_9^{-1}\\
& +0.427Q_FU_9^{-1}+0.855Q_GU_9^{-1}+1.28Q_HU_9^{-1}+1.71Q_IU_9^{-1}+2.14Q_JU_9^{-1}\\
& +0.637Q_AU_{10}^{-1}+1.27Q_BU_{10}^{-1}+1.91Q_CU_{10}^{-1}+2.55Q_DU_{10}^{-1}+3.18Q_EU_{10}^{-1}\\
& +0.637Q_FU_{10}^{-1}+1.27Q_GU_{10}^{-1}+1.91Q_HU_{10}^{-1}+2.55Q_IU_{10}^{-1}+3.18Q_JU_{10}^{-1}\\
& +5824.88S_1U_1^{-1}+5574.07S_2U_2^{-1}+4208.44S_3U_3^{-1}+3507.03S_4U_4^{-1}+9280.88S_5U_5^{-1}\\
& +5288.44S_6U_6^{-1}+4407.03S_7U_7^{-1}+5504.44S_8U_8^{-1}+10254.07S_9U_9^{-1}+10254.07S_{10}U_{10}^{-1}\\
& +44.06Q_A+53.34Q_B+61.84Q_C+69.93Q_D+77.76Q_E\\
& +49.9Q_F+62.02Q_G+73.8Q_H+85.04Q_I+95.94Q_J
\end{aligned}
$$

$s.t.$

$$3.57U_1 + 28.57S_1 Q_A^{-1} + 57.14S_1 Q_B^{-1} + 85.71S_1 Q_C^{-1} + 114.29S_1 Q_D^{-1} + 142.86S_1 Q_E^{-1}$$
$$+ 28.57S_1 Q_F^{-1} + 57.14S_1 Q_G^{-1} + 85.71S_1 Q_H^{-1} + 114.29S_1 Q_I^{-1} + 142.86S_1 Q_J^{-1} \le 1$$

$$2.5U_2 + 20S_2 Q_A^{-1} + 40S_2 Q_B^{-1} + 60S_2 Q_C^{-1} + 80S_2 Q_D^{-1} + 100S_2 Q_E^{-1}$$
$$+ 20S_2 Q_F^{-1} + 40S_2 Q_G^{-1} + 60S_2 Q_H^{-1} + 80S_2 Q_I^{-1} + 100S_2 Q_J^{-1} \le 1$$

$$3.57U_3 + 28.57S_3 Q_F^{-1} + 57.14S_3 Q_G^{-1} + 85.71S_3 Q_H^{-1} + 114.29S_3 Q_I^{-1} + 142.86S_3 Q_J^{-1} \le 1$$

$$2.5U_4 + 20S_4 Q_A^{-1} + 40S_4 Q_B^{-1} + 60S_4 Q_C^{-1} + 80S_4 Q_D^{-1} + 100S_4 Q_E^{-1} \le 1$$

$$3.57U_5 + 28.57S_5 Q_A^{-1} + 57.14S_5 Q_B^{-1} + 85.71S_5 Q_C^{-1} + 114.29S_5 Q_D^{-1} + 142.86S_5 Q_E^{-1}$$
$$+ 28.57S_5 Q_F^{-1} + 57.14S_5 Q_G^{-1} + 85.71S_5 Q_H^{-1} + 114.29S_5 Q_I^{-1} + 142.86S_5 Q_J^{-1} \le 1$$

$$3.57U_6 + 28.57S_6 Q_F^{-1} + 57.14S_6 Q_G^{-1} + 85.71S_6 Q_H^{-1} + 114.29S_6 Q_I^{-1} + 142.86S_6 Q_J^{-1} \le 1$$

$$2.5U_7 + 20S_7 Q_A^{-1} + 40S_7 Q_B^{-1} + 60S_7 Q_C^{-1} + 80S_7 Q_D^{-1} + 100S_7 Q_E^{-1} \le 1$$

$$3.57U_8 + 28.57S_8 Q_F^{-1} + 57.14S_8 Q_G^{-1} + 85.71S_8 Q_H^{-1} + 114.29S_8 Q_I^{-1} + 142.86S_8 Q_J^{-1} \le 1$$

$$2.5U_9 + 20S_9 Q_A^{-1} + 40S_9 Q_B^{-1} + 60S_9 Q_C^{-1} + 80S_9 Q_D^{-1} + 100S_9 Q_E^{-1}$$
$$+ 20S_9 Q_F^{-1} + 40S_9 Q_G^{-1} + 60S_9 Q_H^{-1} + 80S_9 Q_I^{-1} + 100S_9 Q_J^{-1} \le 1$$

$$3.57U_{10} + 28.57S_{10} Q_A^{-1} + 57.14S_{10} Q_B^{-1} + 85.71S_{10} Q_C^{-1} + 114.29S_{10} Q_D^{-1} + 142.86S_{10} Q_E^{-1}$$
$$+ 28.57S_{10} Q_F^{-1} + 57.14S_{10} Q_G^{-1} + 85.71S_{10} Q_H^{-1} + 114.29S_{10} Q_I^{-1} + 142.86S_{10} Q_J^{-1} \le 1$$

$$0.002 \le S_i \le 0.005 \quad for\ i = 1,2,5,9,10$$
$$0.002 \le S_i \le 0.010 \quad for\ i = 3,4,6,7,8$$

$$Q_A \le 8 \qquad Q_B \le 16 \qquad Q_C \le 24 \qquad Q_D \le 3.2 \qquad Q_E \le 40$$
$$Q_F \le 8 \qquad Q_G \le 16 \qquad Q_H \le 24 \qquad Q_I \le 3.2 \qquad Q_J \le 40$$

$$Q_i\ ,\ S_j\ ,\ U_j > 0 \qquad for\ i = A, .., J\ ;\ j = 1, .., 10$$

Optimum solution cost =33317.838

Problem P15 N=22 M=14 D=50

$$
\begin{aligned}
\min\ F = {} & 19.4x_{11}^{-1.47} + 86.0x_{12}^{-0.38} + 152x_{13}^{-0.27} \\
& + 19.4x_{21}^{-1.47} + 16.8x_{22}^{-1.66} + 27.4x_{23}^{-0.63} + 179x_{24}^{-0.37} \\
& + 19.4x_{31}^{-1.47} + 16.8x_{32}^{-1.66} + 91.5x_{33}^{-0.30} + 120x_{34}^{-0.33} \\
& + 19.4x_{41}^{-1.47} + 45.9x_{42}^{-0.45} + 179x_{43}^{-0.37} \\
& + 19.4x_{51}^{-1.47} + 16.8x_{52}^{-1.66} + 91.5x_{53}^{-0.30} + 152x_{54}^{-0.27} \\
& + 19.4x_{61}^{-1.47} + 16.8x_{62}^{-1.66} + 27.4x_{63}^{-0.63} + 179x_{64}^{-0.37}
\end{aligned}
$$

$s.t.$

$$
\begin{aligned}
& 0.7756x_{11}x_{12}x_{13} \le 1 \\
& 0.8544x_{11}x_{12}x_{13} + 0.9172x_{21}x_{22}x_{23}x_{24} \le 1 \\
& 0.8804x_{11}x_{12}x_{13} + 2.0082x_{21}x_{22}x_{23}x_{24} + 0.1029x_{31}x_{32}x_{33}x_{34} \le 1 \\
& 0.8540x_{11}x_{12}x_{13} + 4.3310x_{21}x_{22}x_{23}x_{24} + 0.3305x_{31}x_{32}x_{33}x_{34} + 0.4018x_{41}x_{42}x_{43}x_{44} \le 1 \\
& 0.8433x_{11}x_{12}x_{13} + 4.4959x_{21}x_{22}x_{23}x_{24} + 0.3476x_{31}x_{32}x_{33}x_{34} + 0.4337x_{41}x_{42}x_{43}x_{44} \\
& \quad + 0.0120x_{51}x_{52}x_{53}x_{54} \le 1 \\
& 0.8162x_{11}x_{12}x_{13} + 4.8288x_{21}x_{22}x_{23}x_{24} + 0.3827x_{31}x_{32}x_{33}x_{34} + 0.5003x_{41}x_{42}x_{43}x_{44} \\
& \quad + 0.0379x_{51}x_{52}x_{53}x_{54} + 1.6400x_{61}x_{62}x_{63}x_{64} \le 1 \\
& 0.10x_{11}^{-1}x_{12}^{-1} \le 1 \\
& 0.15x_{21}^{-1}x_{22}^{-1}x_{23}^{-1} \le 1 \\
& 0.15x_{31}^{-1}x_{32}^{-1}x_{33}^{-1} \le 1 \\
& 0.15x_{41}^{-1}x_{42}^{-1} \le 1 \\
& 0.15x_{51}^{-1}x_{52}^{-1}x_{53}^{-1} \le 1 \\
& 0.15x_{61}^{-1}x_{62}^{-1}x_{63}^{-1} \le 1 \\
& 1.4286x_{61}x_{62} \le 1 \\
& 1.25x_{41} \le 1
\end{aligned}
$$

$$
\begin{aligned}
0 \le x_{ij} \le 1 \qquad & i = 1,4\ ;\ j = 1,\ldots,3 \\
& i = 2,3,5,6\ ;\ j = 1,\ldots,4
\end{aligned}
$$

Optimum solution cost = 1831.8068

Problem P16 N=10 M=7 D=9

$$\min \quad 2541x_1 + 0.012293x_7 + 2.419x_3x_4x_6 + 77171x_3^{-1.8}x_5^{-4.8}x_6$$

$s.t.$

$$888.76x_7^{-1}x_8 + 0.23972x_2x_7^{-1}x_8 + x_7^{-1}x_8x_9x_{10} \le 1$$

$$0.62004x_9^{-1} + 1.1072 * 10^{-3}x_2x_9^{-1} \le 1$$

$$2.1872 * 10^{18}x_1^{-2}x_2^{-6.0244}x_{10}^{-1} + 6.2139 * 10^{-8}x_1x_2^{3.0122}x_{10}^{-1} \le 1$$

$$50x_2^{-1} + 4.7394 * 10^{-9}x_2^{-1}x_7 + 7.3124 * 10^{-6}x_2^{-1}x_3^{-7/6}x_4^{-1}x_6^{-4/3}x_7^{4/3} + 1.2577 * 10^{-5}x_2^{-1}x_3^{-0.2}x_5^{0.8}x_6^{-1}x_7 \le 1$$

$$2.447 * 10^{6}x_8^{-1} + 1.0289 * 10^{-3}x_2 \le 1$$

$$8.1667 * 10^{-3}x_4^{-1} + x_4^{-1}x_5 \le 1$$

$$0.083333x_4^{-1} \le 1$$

$$x_i \ge 0 \qquad i = 1, .., 10$$

Optimum solution cost =32373281