Question 1 (Q. 51, p. 122)  
(Solutions to Assignment 3)  

Let  
\[ X_1 = \text{number of transistors' worth of germanium melted by method 1} \]  
\[ X_2 = \text{number of transistors' worth of germanium melted by method 2} \]  
\[ RD = \text{number of defective transistors' worth of germanium refired} \]  
\[ R_1 = \text{number of grade 1 transistors' worth of germanium refired} \]  
\[ R_2 = \text{number of grade 2 transistors' worth of germanium refired} \]  
\[ R_3 = \text{number of grade 3 transistors' worth of germanium refired} \]  
\[ D = \text{number of defective transistors' worth of germanium not refired} \]  
\[ Y_1 = \text{number of grade 1 transistors' worth of germanium not refired} \]  
\[ Y_2 = \text{number of grade 2 transistors' worth of germanium not refired} \]  
\[ Y_3 = \text{number of grade 3 transistors' worth of germanium not refired} \]  

Schematic:
Then the appropriate LP is

Min $z = (50X_1 + 70X_2) + (25RD + 25R1 + 25R2 + 25R3)$

Costs

(Melting)  (Refiring)

s.t.

\[0.3X_1 + 0.2X_2 = RD + D\]  
\[0.3X_1 + 0.2X_2 = R1 + Y1\]  
\[0.2X_1 + 0.25X_2 = R2 + Y2\]  
\[0.15X_1 + 0.20X_2 = R3 + Y3\]  

Material Balance - First Stage (Melting)

\[0.25RD + 0.30R1 + Y1 \geq 3000\]  
\[0.15RD + 0.30R1 + 0.40R2 + Y2 \geq 3000\]  
\[0.20RD + 0.20R1 + 0.30R2 + 0.50R3 + Y3 \geq 2000\]  
\[0.05X_1 + 0.15X_2 + 0.10RD + 0.20R1 + 0.30R2 + 0.50R3 \geq 1000\]  

Grade 4 demand  

Second Stage (Refire)

\[X_1 + X_2 + RD + R1 + R2 + R3 \leq 20000\]  

(capacity)

All variables $\geq 0$

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Let

X1 = Tons of purchased boxboard sent through deinking 
T1 = Tons of purchased tissue sent through deinking 
N1 = Tons of purchased newsprint sent through deinking 
B1 = Tons of purchased book paper sent through deinking 
X2 = Tons of purchased boxboard sent through asphalt dispersion 
T2 = Tons of purchased tissue sent through asphalt dispersion 
N2 = Tons of purchased newsprint sent through asphalt dispersion 
B2 = Tons of purchased book paper sent through asphalt dispersion 
PX = Tons of available boxboard pulp 
PT = Tons of available tissue pulp 
PN = Tons of available newspaper pulp 
PB = Tons of available book paper pulp 
PXi = Tons of boxboard pulp used for grade $i$ paper, $i=2,3$ 
PTi = Tons of tissue pulp used for grade $i$ paper, $i=2,3$ 
PNi = Tons of newspaper pulp used for grade $i$ paper, $i=1,3$ 
PBi = Tons of book paper pulp used for grade $i$ paper, $i=1,2$

A schematic is shown below…
Min Z = \[ 5(X_1 + X_2) + 6(T_1+T_2) + 8(N_1+N_2) + 10(B_1+B_2) \] (raw materials cost) 
\[ + 20(X_1+T_1+N_1+B_1) + 15(X_2+T_2+N_2+B_2) \] (processing cost) 
\[\text{st}\]
\[0.9*0.15*X_1 + 0.8*0.15*X_2 = PX, \text{i.e.} \quad 0.135X_1 + 0.12X_2 - PX = 0\]
\[0.9*0.20*T_1 + 0.8*0.20*T_2 = PT, \text{i.e.} \quad 0.18T_1 + 0.16T_2 - PT = 0\]
\[0.9*0.30*N_1 + 0.8*0.30*N_2 = PN, \text{i.e.} \quad 0.27N_1 + 0.24N_2 - PN = 0\]
\[0.9*0.40*B_1 + 0.8*0.40*B_2 = PB, \text{i.e.} \quad 0.36B_1 + 0.32B_2 - PB = 0\]
(material balance for each pulp type)
\[PX_2 + PX_3 = PX, \text{i.e.} \quad PX_2 + PX_3 - PX = 0\]
\[PT_2 + PT_3 = PT, \text{i.e.} \quad PT_2 + PT_3 - PT = 0\]
\[PN_1 + PN_3 = PN, \text{i.e.} \quad PN_2 + PN_3 - PN = 0\]
\[PB_1 + PB_2 = PB, \text{i.e.} \quad PB_2 + PB_3 - PB = 0\]
(pulp balance; pulp used = pulp produced)
\[PN_1 + PB_1 \geq 500\]
\[PX_2 + PT_2 + PB_2 \geq 500\]
\[PX_3 + PT_3 + PN_3 \geq 600\]
(pulp requirements for paper grades 1, 2, 3)
\[X_1 + T_1 + N_1 + B_1 \leq 3000\]
\[X_2 + T_2 + N_2 + B_2 \leq 3000\]
(processing capacity)

All variables nonnegative

**NOTE:** You could eliminate the pulp balance constraints altogether and also eliminate the four variables PX, PT, PN, PB by replacing these in the first set of constraints with (PX2+PX3), (PT2+PT3), (PN1+PN3), (PB1+PB2), respectively …
**Question 3**

a) Define \( X_j \) = No. of 100 lb. bags of Type \( j \) fertilizer used to blend the order for 1000 lbs of 17-14-10 fertilizer.

Then the LP is

\[
\begin{align*}
\text{Minimize} & \quad 90X_1 + 20X_2 + 30X_3 \\
\text{St} & \quad 100X_1 + 100X_2 + 100X_3 = 1000 \quad \text{(total weight must be 1000 lbs.)} \\
& \quad 50X_1 + 10X_3 \geq 170 \quad \text{(nitrogen content reqmt.)} \\
& \quad 20X_1 + 15X_2 + 10X_3 \geq 140 \quad \text{(phosphorus content reqmt.)} \\
& \quad 5X_1 + 20X_2 + 10X_3 \geq 100 \quad \text{(potassium content reqmt.)} \\
& \quad X_1, X_3 \geq 0 \quad \text{(nonnegativity)}
\end{align*}
\]

b) To convert to a two variable problem we can eliminate one of the variables (say \( X_3 \)) by using the first constraint via \( X_3 = 10 - X_1 - X_2 \):

\[
\begin{align*}
\text{Minimize} & \quad 90X_1 + 20X_2 + 30(10-X_1-X_2) = 60X_1 - 10X_2 + 300, \ i.e., \\
\text{Minimize} & \quad 60X_1 - 10X_2 \\
\text{St} & \quad 50X_1 + 10(10-X_1-X_2) \geq 170 \quad \Rightarrow \quad 40X_1 - 10X_2 \geq 70 \\
& \quad 20X_1 + 15X_2 + 10(10-X_1-X_2) \geq 140 \quad \Rightarrow \quad 10X_1 + 5X_2 \geq 40 \\
& \quad 5X_1 + 20X_2 + 10(10-X_1-X_2) \geq 100 \quad \Rightarrow \quad -5X_1 + 10X_2 \geq 0 \\
& \quad X_1, X_2, (10-X_1-X_2) \geq 0 \quad \Rightarrow \quad X_1 + X_2 \leq 10, \ X_1, X_2 \geq 0
\end{align*}
\]

**Note:** The value of 300 is a constant that may be factored out of the objective for the modified problem; it just needs to be added on to the optimum value of the modified problem above to get the actual cost for the original.

**Isocost line for part (c)**

**Isocost line for part (d)**

**Extreme Points**

- A \((3.2, 1.6)\)
- B \((2.5, 3)\)
- C \((3.4, 6.6)\)
- D \((6.67, 3.33)\)
c) From the graph and the slope of the isocost line shown, it is clear that the optimum is at the extreme point B, with $X_1=2.5$, $X_2=3$. Note that this implies that $X_3 = 10 - (2.5+3) = 4.5$ bags. The minimum cost is given by $(60*2.5) - (10*3) + 300 = 420$, or equivalently, $(90*2.5) + (20*3) + (30*4.5) = \$420$. The constraints on nitrogen and phosphorus are active, and the bag and potassium constraints are inactive (note though, that in terms of the original problem exactly ten bags are used). The minimum nutrient requirements of nitrogen and phosphorus are thus both exactly met, while that of potassium is exceed by $[-(5*2.5) + (10*3)] - 0 = 17.5$ lbs. (or equivalently, by $[(5*2.5) + (20*3) + (10*4.5)] - 100 = 17.5$ lbs.).

d) With the new costs the objective is given by $85X_1 + 10X_2 + 25(10-X_1-X_2) = 60X_1 - 15X_2 + 250$, i.e., Minimize $60X_1 - 15X_2$. From the slope of the isocost line shown, the optimum is along the line joining extreme points B and C. There are two optimum extreme points: $(X_1=2.5, X_2=3)$ and $(X_1=3.4, X_2=6.6)$. The minimum cost in both cases is given by $355 = (60*2.5-15*3+250 = 60*3.4-15*6.6+250)$. There are infinitely many optimum solutions to the LP and are given in general by $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} 2.5 \\ 3 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 3.4 \\ 6.6 \end{bmatrix} = \begin{bmatrix} 3.4-0.9\lambda \\ 6.6-3.6\lambda \end{bmatrix},$$ where $\lambda$ is a constant in the closed interval $[0,1]$. The objective is given by $60*(3.4-0.9\lambda) - 15*(6.6-3.6\lambda) + 250 = 355$. In terms of the original problem, we may specify the complete solution set: $X_1, X_2$ as above and $X_3=10-(X_1+X_2) = 10-[(3.4-0.9\lambda)+(6.6-3.6\lambda)] = 4.5\lambda$, so that the objective is $85*(3.4-0.9\lambda) + 10*(6.6-3.6\lambda) + 25*4.5\lambda = 355$. The constraint on nitrogen is always active and if one chooses B then the phosphorus constraint is also active, while if one chooses C then the bag constraint is also active. The potassium constraint is inactive. The minimum nutrient requirement of nitrogen is exactly met. At B the phosphorus requirement is also exactly met while that of potassium is exceed by $[(5*2.5) + (10*3)] - 0 = 17.5$ lbs. At C the phosphorus requirement is exceeded by $[(10*3.4)+(5*6.6)] - 40 = 27$, while that of potassium is exceed by $[-(5*3.4) + (10*6.6)] - 0 = 49$ lbs. At points between B and C the requirement of phosphorus is exceeded by $[10*(3.4-0.9\lambda) + 5*(6.6-3.6\lambda)] - 40 = (27-27\lambda)$, while that of potassium is exceed by $[-5*(3.4-0.9\lambda) + 10*(6.6-3.6\lambda)] - 0 = (49-31.5\lambda)$ lbs.

**Question 3**

- **Increase**
- **Decrease**

![Graph showing isocost line and constraints](image)
The feasible region is open and unbounded.

The problem has no optimal solution for minimizing $-2X_1 + 6X_2$, since the slope of the isocost line shown is such that it can be moved in the direction of decrease without ever leaving the feasible region completely: $X_1$ can be made arbitrarily large while keeping $X_2$ fixed at some small nonnegative value so that the objective goes to $-\infty$.

If the problem is a maximization of the objective, the problem is still unbounded because the isocost line can also be moved in the direction of increase without ever leaving the feasible region completely: $X_2$ can be made arbitrarily large while ensuring that $6X_2$ is always larger than $2X_1$ so that the objective goes to $\infty$. 