I.E. 2001 OPERATIONS RESEARCH

(Spring 2018: Solutions to Assignment 4)

Question I.

First, subtract a nonnegative excess variable X_5 and add a nonnegative slack variable X_6 to the second constraint to make all constraints equalities. Then rearrange everything so that all the variables are to the left and the constants to the right of the equations and multiply both sides of equations 2 and 3 by the constant -1 so that the RHS is a nonnegative constant for each. These two steps yield:

$$
\begin{array}{lrcl}
\text{Max} & -3X_1 + X_2 - 2X_3 + X_4 & & \\
\text{st} & -4X_1 + X_2 + X_3 \quad - X_5 & = & 4 \\
 & -3X_1 + X_2 - 2X_3 \quad - X_6 & = & 6 \\
 & - X_2 - 4X_3 + X_4 & = & 1 \\
 & 2X_1 - X_2 + X_3 & = & 0 \\
 & X_1 \text{ UNR, } X_2, X_3, X_4, X_5, X_6 \geq 0 & &
\end{array}
$$

Next make sure that all variables are nonnegative by defining $X_1 = X_1' - X_1''$ where $X_1' \geq 0$ and $X_1'' \geq 0$; and obtain the standard form:

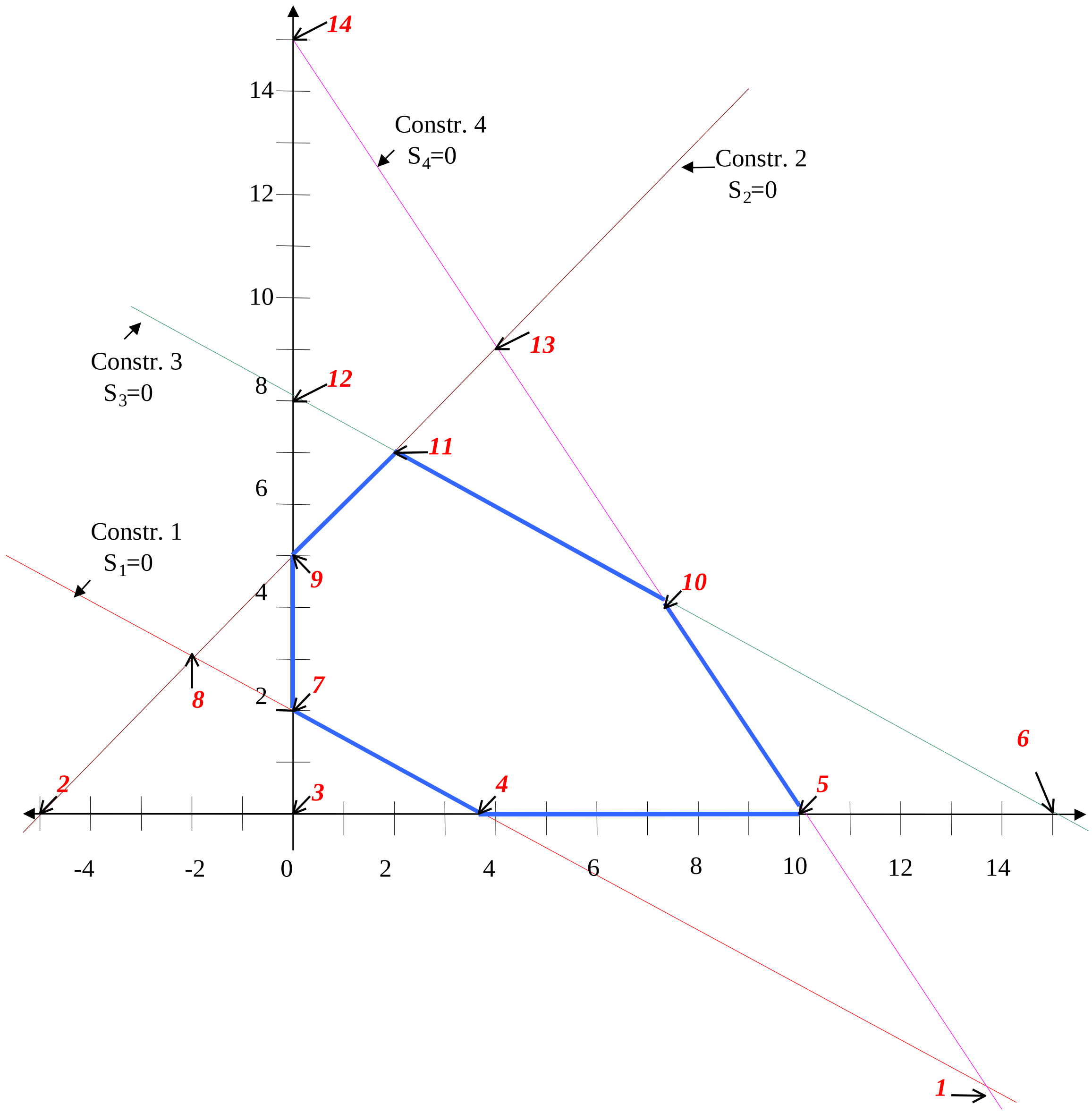
$$
\begin{array}{lrcl}
\text{Max} & -3X_1' + 3X_1'' + X_2 - 2X_3 + X_4 & & \\
\text{st} & -4X_1' + 4X_1'' + X_2 + X_3 \quad - X_5 & = & 4 \\
 & -3X_1' + 3X_1'' + X_2 - 2X_3 \quad - X_6 & = & 6 \\
 & - X_2 - 4X_3 + X_4 & = & 1 \\
 & 2X_1' - 2X_1'' - X_2 + X_3 & = & 0 \\
 & X_1', X_1'', X_2, X_3, X_4, X_5, X_6 \geq 0 & &
\end{array}
$$

Question II.

a) The graph is shown on the next page

b) The standard form yields the following augmented system:

$$
\begin{array}{rcl}
8X_1 + 15X_2 - S_1 & = & 30 \\
-X_1 + X_2 \quad + S_2 & = & 5 \\
2X_1 + 3.75X_2 \quad + S_3 & = & 30 \\
1.5X_1 + X_2 \quad + S_4 & = & 15
\end{array}
$$



The maximum no. of basic solutions that could possibly exist is

$\binom{n}{n-m} = \binom{n}{m} = \binom{6}{4} = 15$. Only 14 of these exist here (as shown above in red); there is no basic solution corresponding to (S_1, S_3) being nonbasic (notice that the slopes of constraints 1 and 3 are identical and the lines corresponding to these do not intersect).

c) The problem has six basic feasible solutions.

d) The basic solutions are marked on the graph, the ones marked **4**, **5**, **7**, **9**, **10** and **11** are also basic *feasible* solution**.** The variable values at the fourteen basic solutions (found by setting the two nonbasic variables to zero and solving the resultant 4×4 system of equations) are as follows (Basic **feasible** solutions are in boldface and red):

Point	Nonbasic Variables	Basic Variables
1	$S_1=S_4=0$	X_1=13.45, X_2=-5.17, S_2=23.62, S_3=22.5
2	$S_2=X_2=0$	X_1=-5, S_1=-70, S_3=40, S_4=22.5
3	$X_1=X_2=0$	S_1=-30, S_2=5, S_3=30, S_4=15
4	$S_1=X_2=0$	**X_1=3.75, S_2=8.75, S_3=22.5, S_4=9.375**
5	$S_4=X_2=0$	**X_1=10, S_1=50, S_2=15, S_3=10**
6	$S_3=X_2=0$	X_1=15, S_1=90, S_2=20, S_4=-7.5
7	$S_1=X_1=0$	**X_2=2, S_2=3, S_3=22.5, S_4=13**
8	$S_1=S_2=0$	X_1=-1.96, X_2=3.04, S_3=22.52, S_4=14.9
9	$S_2=X_1=0$	**X_2=5, S_1=45, S_3=11.25, S_4=10**
10	$S_3=S_4=0$	**X_1=7.24, X_2=4.14, S_1=90, S_2=8.1**
11	$S_2=S_3=0$	**X_1=1.96, X_2=6.96, S_1=90, S_4=5.11**
12	$S_3=X_1=0$	X_2=8, S_1=90, S_2=-3, S_4=7
13	$S_4=S_2=0$	X_1=4, X_2=9, S_1=137, S_3=-11.75
14	$S_4=X_1=0$	X_2=15, S_1=195, S_2=-10, S_3=-26.25

e)

Basic	Eq.	Z	X_1	X_2	S_1 ↓	S_2	S_3	S_4	RHS	
Z	(0)	1	0	7.25	-0.75	0	0	0	22.5	
X_1	(1)	0	1	1.875	-0.125	0	0	0	3.75	-
S_2	(2)	0	0	2.875	-0.125	1	0	0	8.75	-
S_3	(3)	0	0	0	0.25	0	1	0	22.5	22.5/0.25
→ S_4	(4)	0	0	-1.8125	**0.1875**	0	0	1	9.375	9.375/0.1875

Note that the above tableau corresponds to BFS No. 4.
S_1 is the entering variable and S_4 is the leaving variable as it wins the ratio test.
Now perform the following sequence of *ero*'s to get a canonical form once again:

1) Row4 ← Row4/0.1875
2) Row0 ← Row0+0.75Row4
3) Row1 ← Row1+0.125Row4
4) Row2 ← Row2+0.125Row4
5) Row3 ← Row3-0.25Row4

Basic	Eq.	Z	X_1	X_2	S_1	S_2	S_3	S_4	RHS
Z	(0)	1	0	0	0	0	0	4	60
X_1	(1)	0	1	0.6667	0	0	0	0.667	10
S_2	(2)	0	0	1.6667	0	1	0	0.667	15
S_3	(3)	0	0	2.4167	0	0	1	-1.333	10
S_1	(4)	0	0	-9.667	1	0	0	5.333	50

For a maximization problem, this is the optimal tableau since none of the reduced costs in Eq. 0 are negative, i.e., no further improvement is possible - the optimum value of the objective is equal to 60. The solution obtained is (X_1=10, X_2=0, S_1=50, S_2=15, S_3=10) which corresponds to BFS #5. However, one unusual thing is that that the reduced cost for X_2 (entry in Row 0) is **equal** to 0; usually, all nonbasic variables have positive reduced costs at the optimum. This indicates alternative optima. If we now enter X_1 into the basis (and remove S_3 by the minimum ratio test) then we get a new optimum basis

corresponding to BFS #10 with (X_1=7.24, X_2=4.14, S_1=90, S_2=8.1, S_3=0, S_4=0) and the same value of 60 for the objective.
NOTE: In general any convex combination (weighted average) of these two solutions will also be an optimum solution yielding the same value of 60 for the objective, and the entire optimum solution set may be specified as:

$$\begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} = \lambda \begin{bmatrix} 10 \\ 0 \\ 50 \\ 15 \\ 10 \\ 0 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 7.24 \\ 4.14 \\ 90 \\ 8.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7.24-2.76\lambda \\ 4.14-4.14\lambda \\ 90-40\lambda \\ 8.1-6.9\lambda \\ 10\lambda \\ 0 \end{bmatrix} \quad where\ 0 \le \lambda \le 1.$$

Question III. (Remember that this is a minimization problem!)

ITERATION 1

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS	
Z	(0)	1	3	-1	3	4	0	0	0	0	
S_1	(2)	0	1	7	3	7	1	0	0	46	46/7
S_2	(3)	0	3	-1	1	**2**	0	1	0	8	***8/2***
S_3	(4)	0	2	3	-1	1	0	0	1	10	10/1

ITERATION 2

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS	
Z	(0)	1	-3	1	1	0	0	-2	0	-16	
S_1	(2)	0	-9.5	10.5	-0.5	0	1	-3.5	0	18	∞
X_4	(3)	0	1.5	-0.5	**0.5**	1	0	0.5	0	4	***1/0.5***
S_3	(4)	0	0.5	3.5	-1.5	0	0	-0.5	1	6	∞

(could also have chosen X_2 to enter – break ties arbitrarily)

ITERATION 3

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS	
Z	(0)	1	-6	2	0	-2	0	-3	0	-24	
S_1	(2)	0	-8	**10**	0	1	1	-3	0	22	***22/10***
X_3	(3)	0	3	-1	1	2	0	1	0	8	∞
S_3	(4)	0	5	2	0	3	0	1	1	18	18/2

ITERATION 4

Basic	Eq.	Z	X_1	X_2	X_3	X_4	S_1	S_2	S_3	RHS
Z	(0)	1	-4.4	0	0	-2.2	-0.2	-2.4	0	-28.4
X_2	(2)	0	-0.8	1	0	0.1	0.1	-0.3	0	2.2
X_3	(3)	0	2.2	0	1	2.1	0.1	0.7	0	10.2
S_3	(4)	0	6.6	0	0	2.8	-0.2	1.6	1	13.6

OPTIMUM SOLUTION!