I.E. 2001 OPERATIONS RESEARCH

(Solutions to Assignments 7)

Question 1

1) Minimize $Z = 4x_1 + 3x_2$

$$\begin{aligned} \text{st} \quad & 2x_1 + x_2 \geq 25 \\ & -3x_1 + 2x_2 \geq 15 \\ & x_1 + x_2 \geq 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

All inequality constraints are "normal" (≥ for a Min problem) so we can proceed...

The dual is

Maximize $W = 25y_1 + 15y_2 + 15y_3$

$$\begin{aligned} \text{st} \quad & 2y_1 - 3y_2 + y_3 \leq 4 \\ & y_1 + 2y_2 + y_3 \leq 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

2)

Maximize $Z = -2x_1 + x_2 - 4x_3 + 3x_4$

$$\begin{aligned} \text{st} \quad & x_1 + x_2 + 3x_3 + 2x_4 \geq 10 \\ & x_1 + x_2 + 3x_3 + 2x_4 \leq 40 \\ & -x_1 + x_3 - x_4 \leq 10 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + 2x_2 + x_3 + 2x_4 = 20 \\ & x_2, x_3, x_4 \geq 0; \; x_1 \text{ UNR.} \end{aligned}$$

Before finding the dual ensure that all inequality constraints are "normal" (i.e., ≤ for a Max problem)

Maximize $Z = -2x_1 + x_2 - 4x_3 + 3x_4$

$$\begin{aligned} \text{st} \quad & -x_1 - x_2 - 3x_3 - 2x_4 \leq -10 \\ & x_1 + x_2 + 3x_3 + 2x_4 \leq 40 \\ & -x_1 + x_3 - x_4 \leq 10 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + 2x_2 + x_3 + 2x_4 = 20 \\ & x_2, x_3, x_4 \geq 0; \; x_1 \text{ UNR.} \end{aligned}$$

The dual is

Minimize $W = -10y_1 + 40y_2 + 10y_3 + 20y_4 + 20y_5$

st $-y_1 + y_2 - y_3 + 2y_4 + y_5 = -2$ ("=" since it corresponds to a UNR variable x_1)

$-y_1 + y_2 + y_4 + 2y_5 \geq 1$

$-3y_1 + 3y_2 + y_3 + y_5 \geq -4$

$-2y_1 + 2y_2 - y_3 + 2y_5 \geq 3$

$y_1, y_2, y_3, y_4 \geq 0$, y_5 UNR. ("UNR" since it corresponds to an "=" constr.)

Question 2

i)

Maximize	$Z = 3x_1 + 2x_2$	Dual is	Minimize	$W = 20y_1 + 16y_2$
st	$5x_1 + 4x_2 \le 20$		st	$5y_1 + 2y_2 \ge 3$
	$2x_1 + 4x_2 \le 16$			$4y_1 + 4y_2 \ge 2$
	$x_1, x_2 \ge 0$			$y_1, y_2 \ge 0$

The optimum dual solution is y_1*=0.6, y_2*=0, W*=12 (from LINDO). By the strong duality theorem, since the dual is feasible and has a finite optimum, so does the original (primal). Moreover, the optimum value of the primal will also be equal to12.

ii)

Minimize	$Z = -3x_1 + 4x_2$	Dual is	Maximize	$W = 2y_1 - 3y_2$
st	$-x_1 + x_2 \ge 2$		st	$-y_1 + y_2 \le -3$
	$-x_1 - 2x_2 \le 3$ (i.e., $x_1 + 2x_2 \ge -3$)			$y_1 + 2y_2 \le 4$
	$x_1, x_2 \ge 0$			$y_1, y_2 \ge 0$

When we attempt to solve the dual we find that its optimum value is 8. Therefore as a consequence of the strong duality theorem, the original (primal) problem also has the same optimal value.

iii)

Maximize	$Z = x_1 + x_2$	Dual is	Minimize	$W = -y_1 - y_2$
st	$-x_1 + x_2 \ge 1$ ($x_1 - x_2 \le -1$)		st	$y_1 - y_2 \ge 1$
	$x_1 - x_2 \ge 1$ ($-x_1 + x_2 \le -1$)			$-y_1 + y_2 \ge 1$
	$x_1, x_2 \ge 0$			$y_1, y_2 \ge 0$

When we attempt to solve the dual we find that it is infeasible. Therefore, the original (primal) problem is **either infeasible or unbounded** and has no optimum solution.

Question 3

1. Since the total supply (=580) is less than total demand (=610), add a dummy source (*i=4)* to "supply" the excess demand. Define X_{ij} as the no. of CWT shipped from plant *i* to warehouse *j* for *i=1,2,3,4* and *j=1,2,3,4,5,* C_{ij} as the cost to ship from plant *i* to warehouse *j* for *i=1,2,3,4* and *j=1,2,3,4,5.* The LP is:

Min $4X_{11}+5X_{12}+...+6X_{34}+10X_{35}+0.15X_{41}+0.10X_{42}+0.25X_{43}+0.20X_{44}+0.05X_{45}$

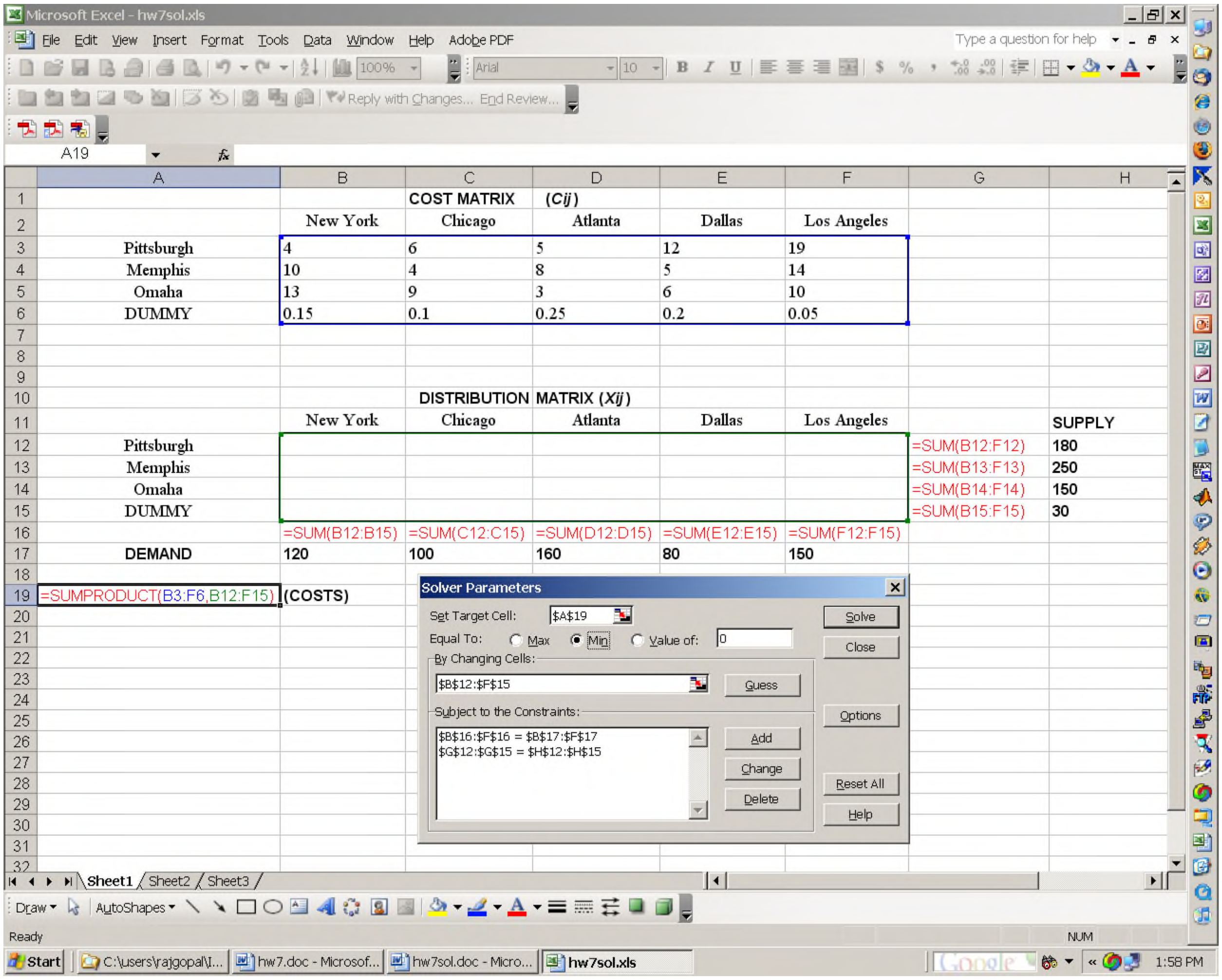
st ← (from dummy source) →

$\sum_j X_{1j}=180$, $\sum_j X_{2j}=250$, $\sum_j X_{3j}=150$, $\sum_j X_{4j}=30$

$\sum_i X_{i1}=120$, $\sum_i X_{i2}=100$, $\sum_i X_{i3}=160$, $\sum_i X_{i4}=80$, $\sum_i X_{i5}=150$

All X_{ij} nonnengative.

The Excel worksheet with the cost-requirements tableau for this is as shown below.



Solving the problem using `Excel-Solver` yields the following optimal shipping plan, with cost = **$3361.50**

	NY	Chi	Atl	Dal	LA	SUPPLY
Pgh	120	0	60	0	0	180
Memphis	0	100	0	80	70	250
Omaha	0	0	100	0	50	150
Dummy	*0*	*0*	*0*	*0*	*30*	*30*
DEMAND	120	100	160	80	150	

The optimal plan calls for shipments as shown above. Note that the 30 units shipped from plant 4 (the Dummy) to Los Angeles signify that Los Angeles has an unsatisfied demand of 30 CWT since plant 4 is a fictitious source that does not actually exist - each plant ships out all available supply at the plant.

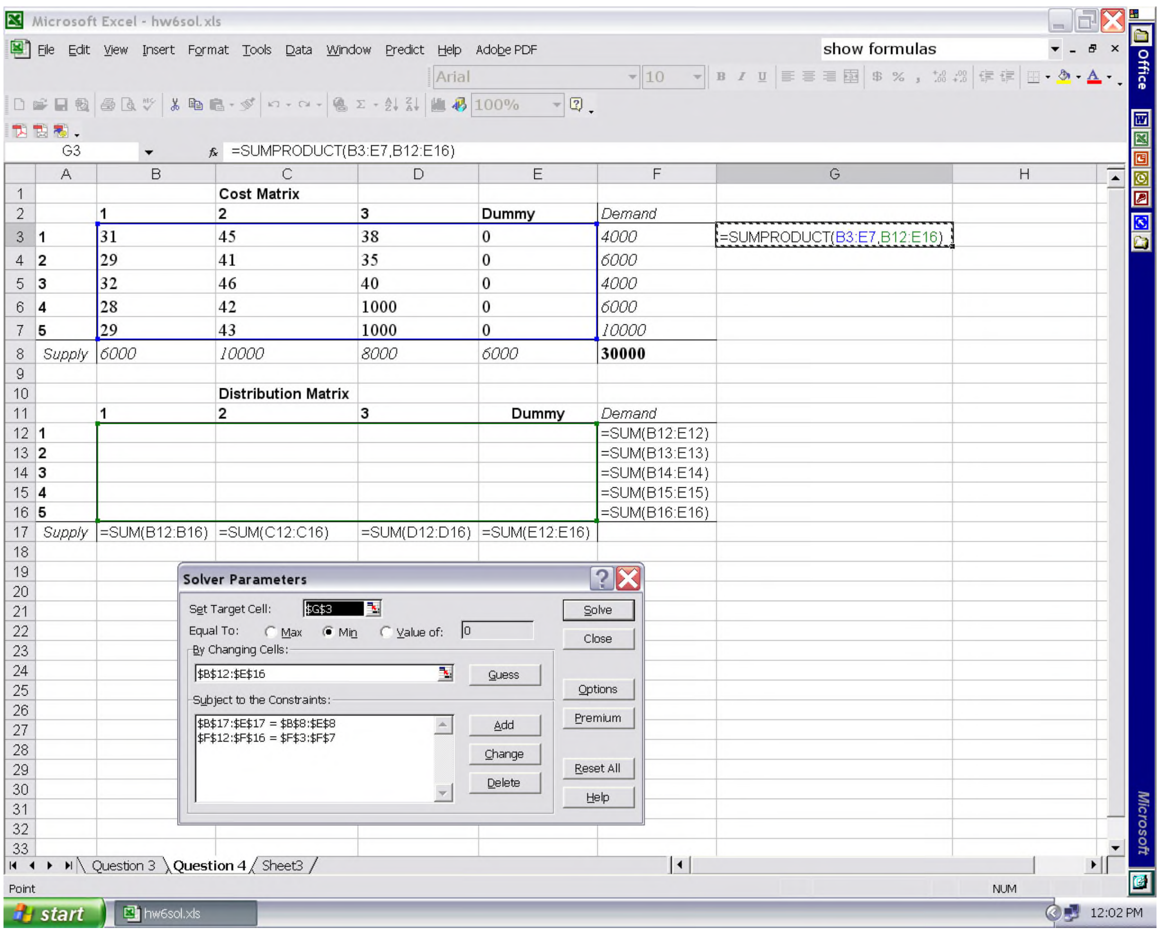
2. If supply at Pittsburgh increases to 230 CWT then total supply = 630, which now exceeds total demand (=610). So we add a dummy destination ($j=6$) to "absorb" the excess supply. Define X_{ij} as the no. of CWT shipped from plant i to warehouse j for $i=1,2,3$ and $j=1,2,3,4,5,6$, and C_{ij} as the cost to *produce and* ship from plant i to warehouse j for $i=1,2,3$ and $j=1,2,3,4,5,6$. The LP is:

Question 4

Define X_{ij} = no. of units of product *j* made at plant *i*. The cost and requirements table is shown below:

	1	2	3	4 (Dummy)	Supply
1	31	45	38	0	*4,000*
2	29	41	35	0	*6,000*
3	32	46	40	0	*4,000*
4	28	42	M	0	*6,000*
5	29	43	M	0	*10,000*
Demand	*6000*	*10,000*	*8,000*	*6,000*	**30,000**

The Excel worksheet is shown below:



The resulting optimal plan is as follows:

	1	2	3	4 (Dummy)	Supply
1	0	0	2000	2000	*4,000*
2	0	0	6000	0	*6,000*
3	0	0	0	4000	*4,000*
4	6000	0	0	0	*6,000*
5	0	10000	0	0	*10,000*
Demand	*6000*	*10,000*	*8,000*	*6,000*	**30,000**

The optimal value of the objective, ***Z*=$884,000**. Note that the problem has multiple optima (for instance an alternative solution has X_{51}=6000 X_{42}=6000, X_{52}=4000; other values are the same).

Question 5:

The cost and requirements matrix is identical to the tableau given, with the exception that we have:

(1) cost of M for all paths that do not exist and 0 for all paths from a node to itself,
(2) supplies of 300 at all junctions and warehouses and 375, 425 and 400 at the three canneries, and
(3) demands of 300 at all canneries and junctions and 380, 365, 370 and 385 at the four warehouses.

The solution with a cost of $145,175 is shown below:

	C1	C2	C3	J1	J2	J3	J4	J5	W1	W2	W3	W4		
C1	300	0	0	0	**75**	0	0	0	0	0	0	0	*375*	***75***
C2	0	300	0	0	0	0	0	0	**80**	**45**	0	0	*425*	***125***
C3	0	0	300	0	0	0	0	**30**	0	0	**70**	0	*400*	***100***
J1	0	0	0	300	0	0	0	0	0	0	0	0	*300*	
J2	0	0	0	0	225	0	0	0	0	**75**	0	0	*300*	
J3	0	0	0	0	0	300	0	0	0	0	0	0	*300*	
J4	0	0	0	0	0	0	300	0	0	0	0	0	*300*	
J5	0	0	0	0	0	0	0	270	0	0	0	**30**	*300*	
W1	0	0	0	0	0	0	0	0	300	0	0	0	*300*	
W2	0	0	0	0	0	0	0	0	0	245	0	**55**	*300*	
W3	0	0	0	0	0	0	0	0	0	0	300	0	*300*	
W4	0	0	0	0	0	0	0	0	0	0	0	300	*300*	
	300	*300*	*300*	*300*	*300*	*300*	*300*	*300*	*380*	*365*	*370*	*385*		
									80	***65***	***70***	***85***		**300**