Question 1 (Q1, p. 502)
Recall that in HW 8 we had defined
Define $x_j = 1$ if player $j$ starts; $j=1,2,3,4,5,6,7$
$0$ otherwise.

We have the following additional constraints now:

1. $x_3 + x_6 \leq 1$
   (if 3 starts, 6 cannot; i.e., $x_3=1 \Rightarrow x_6=0$)

2.1 $x_1 \leq My$
2.2 $2-x_4-x_5 \leq M(1-y)$
   (if $x_1=1$, then $x_4$, $x_5$ are both $1$, i.e., if $x_1>0 \Rightarrow x_4+x_5 \geq 2$, i.e, $2-x_4-x_5 \leq 0$
   This gets us into the format of “$g_i(x)>0 \Rightarrow g_i(x) \leq 0$” to use the “trick” we saw in class, where $g_i(x)=x_1$ and $g_i(x)=2-x_4-x_5$.
   Thus both $x_1>0$ and $2-x_4-x_5>0$ is impossible, i.e, at least one of $x_1 \leq 0$ and $2-x_4-x_5 \leq 0$ must hold.

So we define $y=0$ or $1$

Then (2.1) states that if $y=0$ then $x_1 \leq 0$, i.e., $x_1=0$ (and we don’t care about $x_4$, $x_5$…)
while (10) states that if $y=1$ then $2-x_4-x_5 \leq 0$, i.e., $x_4+x_5 \geq 2$, i.e., $x_4=x_5=1$

Alternatively, we could also express this constraint via

2. $x_4 \geq x_1$, $x_5 \geq x_1$ or 2. $x_4+x_5 \geq 2x_1$

Question 2 (Q14, p. 503-504)

Define $y_j = 1$ if disk $j$ is stored, 0 otherwise; $j=1,2,\ldots,10$

Part (1)

Then the formulation is

Minimize $Z = 3y_1 + 5y_2 + y_3 + 2y_4 + y_5 + 4y_6 + 3y_7 + y_8 + 2y_9 + 2y_{10}$
   (total storage)
st
$y_1 + y_2 + y_4 + y_5 + y_8 + y_9 \geq 1$ (file 1 must be stored)
$y_1 + y_3 \geq 1$ (file 2 must be stored)
$y_2 + y_3 + y_7 + y_{10} \geq 1$ (file 3 must be stored)
$y_3 + y_6 + y_8 \geq 1$ (file 4 must be stored)
$y_1 + y_2 + y_4 + y_6 + y_7 + y_9 + y_{10} \geq 1$ (file 5 must be stored)

All $y_j = 0$ or $1$.

Note: This is an example of a set covering problem

Excel yielded the optimum solution $y_3=y_5=y_{10}=1$ and all other $y_j=0$, $Z^*=4$
Part (2)

Here we want that if either \( y_3 \) or \( y_5 \) (or both) are equal to 1, then \( y_2 \) MUST be 1
So we could add \( y_2 \geq y_3 \) and \( y_2 \geq y_5 \) to the problem constraints.

With the additional constraint the optimum solution has \( y_1 = y_5 = y_8 = 1 \) and all other \( y_j = 0 \), \( Z^* = 5 \)

Question 3 (Q43, p511)
First note that the fixed and variable costs are as follows:

<table>
<thead>
<tr>
<th>Plant</th>
<th>Fixed Cost ($ Billion)</th>
<th>Variable Costs ($ Billion)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Taurus</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>7.5</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8.5</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>9.5</td>
</tr>
</tbody>
</table>

Define \( x_{ij} = 1 \) if Plant \( i (i=1,2,3,4) \) is used to produce type \( j (j=1,2,3) \) cars (Type 1 = Taurus, etc.); 0 otherwise. Also define \( y_i = 1 \) if Plant \( i \) is used to produce any type of car; 0 otherwise (note that one plant will definitely not be used). Then expressing the objective in billions of dollars we have

Min \( 7y_1 + 6y_2 + 4y_3 + 2y_4 + 6x_{11} + 8x_{12} + 4.5x_{13} + 7.5x_{21} + 9x_{22} + 5.5x_{23} + 8.5x_{31} + 9.5x_{32} + 6x_{33} + 9.5x_{41} + 11x_{42} + 7x_{43} \)

st

A.
- \( x_{11} + x_{12} + x_{13} \leq 1 \) (Plant 1 produces no more than one car type)
- \( x_{21} + x_{22} + x_{23} \leq 1 \) (Plant 2 produces no more than one car type)
- \( x_{31} + x_{32} + x_{33} \leq 1 \) (Plant 3 produces no more than one car type)
- \( x_{41} + x_{42} + x_{43} \leq 1 \) (Plant 4 produces no more than one car type)

B.
- \( x_{11} + x_{21} + x_{31} + x_{41} = 1 \) (Car type 1 is produced at exactly one plant)
- \( x_{12} + x_{22} + x_{32} + x_{42} = 1 \) (Car type 2 is produced at exactly one plant)
- \( x_{13} + x_{23} + x_{33} + x_{43} = 1 \) (Car type 3 is produced at exactly one plant)

C.
- \( x_{11} \leq y_{11} \), \( x_{12} \leq y_{12} \), \( x_{13} \leq y_{13} \) (If Plant 1 is not used at all, then it produces nothing)
- \( x_{21} \leq y_{21} \), \( x_{22} \leq y_{22} \), \( x_{23} \leq y_{23} \) (If Plant 2 is not used at all, then it produces nothing)
- \( x_{31} \leq y_{31} \), \( x_{32} \leq y_{32} \), \( x_{33} \leq y_{33} \) (If Plant 3 is not used at all, then it produces nothing)
- \( x_{41} \leq y_{41} \), \( x_{42} \leq y_{42} \), \( x_{43} \leq y_{43} \) (If Plant 4 is not used at all, then it produces nothing)

NOTE: You could also combine A and C via \( x_{11} + x_{12} + x_{13} \leq y_{11} \), \( x_{12} + x_{22} + x_{23} \leq y_{22} \) etc.

D.
- \( y_3 + y_4 - 1 \leq M(1-z) \)
- \( 1 - y_j \leq M \)
The last two constraint arises from “If 3 and 4 are used then 1 must be used.” That is, “If \( y_3 + y_4 \geq 2 \), then \( y_1 \geq 1 \),”
  i.e., if \( y_3 + y_4 > 1 \) then \( y_1 \geq 1 \),
  i.e., both \( y_3 + y_4 > 1 \) and \( y_1 < 1 \) is not possible,
  i.e., at least one of \( y_3 + y_4 \leq 1 \) or \( 1 - y_1 \leq 0 \) must hold.
This yields the last two constraints.
Alternatively, we could also express this last condition via $y_1 \geq y_3 + y_4 - 1$ (so if both $y_3$ & $y_4$ are $= 1$, then the RHS is $= 1$ so that $y_1$ is also $= 1$, if not the RHS is $= 0$ or $-1$, so that $y_1$ could be $0$ or $1$)

All variables $\in (0,1)$

Alternatively (and somewhat more complex...) Formulation:
We could also dispense with the $y_i$ variables altogether by pulling in the fixed cost for each plant in each year into the operating costs and altering the last two sets of constraints as follows:

Min $13x_{11} + 15x_{12} + 11.5x_{13} + 13.5x_{21} + 15x_{22} + 11.5x_{23} + 12.5x_{31} + 13.5x_{32} + 10x_{33} + 11.5x_{41} + 13x_{42} + 9x_{43}$

st
Same constraints as in A. and B. above, and replace C. and D. with

- $x_{31} + x_{32} + x_{33} + x_{41} + x_{42} + x_{43} - 1 \leq Mz$
- $x_{11} + x_{12} + x_{13} - 1 \geq -M(1-z)$

All variables $= 0$ or $1$.

Note that A. and B. ensure that exactly 3 of the four plants are used to produce 3 different cars; so the fixed cost will definitely be counted in the objective only once for each of the 3 plants that are used.

The last set of constraints simply ensure that if one of $(x_{31}, x_{12}, x_{33})$ and one of $(x_{41}, x_{42}, x_{43})$ are equal to $1$ (note that the constraints in A preclude more than one in each set from being equal to $1$...), then one of $(x_{11}, x_{12}, x_{13})$ is also equal to $1$.

The optimum solution has $y_1 = y_3 = y_4 = 1$; $x_{11} = x_{32} = x_{43} = 1$ with an objective function value $= 35.5$

Question 4
Define $y_j = 1$ if location $j$ is selected and $0$ otherwise, $j=1,2,...,M$
and $x_{ij} = 1$ if district $i$ is assigned to the firehouse at location $j$, $i=1,2,...,N$, $j=1,2,...,M$

The constraints are as follows:

1) $\sum_{j=1}^{M} x_{ij} = 1, \quad i = 1,2,...,N$ (every district is assigned to exactly one firehouse)

2) $\sum_{i=1}^{N} x_{ij} \leq y_j N, \quad j = 1,2,...,M$ (no district is assigned to an unused location)

(note that the sum of the $x_{ij}$ over all $i$ can obviously never exceed $N$)
3) \[ \sum_{i=1}^{N} p_i x_{ij} = s_j, \quad j = 1, 2, \ldots, M \quad \text{(total population served by location } j) \]

4) \[ \sum_{j=1}^{M} f_j(s_j) \leq B \quad \text{(budgetary constraint)} \]

(Note that 3 and 4 could be combined and } s_j \text{ eliminated...)

5) \[ y_1 + y_2 \geq 2 z_1 \]

6) \[ y_3 + y_4 \geq 2 z_2 \]

7) \[ z_1 + z_2 = 1 \quad \text{(or } \geq 1) \]

(5, 6 and 7 together ensure that at least one of } y_1 + y_2 \geq 2 \text{ and } y_3 + y_4 \geq 2 \text{ is satisfied)

Suppose we define } d_i \text{ as the distance to district } i \text{ from its assigned firehouse and } D=\text{Max}(d_i)

8) \[ d_i = \sum_{j=1}^{M} d_{ij} x_{ij} \quad \text{(CASE 1)} \]

8) \[ D \geq \sum_{j=1}^{M} d_{ij} x_{ij}, \quad i = 1, 2, \ldots, N \quad \text{(CASE 2)} \]

9) \[ x_{ij}, y_j, z_i, z_2 \text{ all 0 or 1 for } i=1,2,\ldots,N, j=1,2,\ldots,M. \]

The objective is to (case 1) \textbf{Minimize } (\Sigma d_i)/N, \text{ or (case 2) \textbf{Minimize } D}