I.E. 2001 OPERATIONS RESEARCH

(Solutions to Assignment 9)

Question 1 *(Q1, p. 502)*

Recall that in HW 8 we had defined

Define x_j = 1 if player j starts; j=1,2,3,4,5,6,7

0 otherwise.

We have the following additional constraints now:

1. $x_3 + x_6 \leq 1$ (if 3 starts, 6 cannot; i.e., $x_3=1 \Rightarrow x_6=0$)

2.1 $x_1 \leq My$

2.2 $2-x_4-x_5 \leq M(1-y)$

(if 1 starts, 4 and 5 must start. Note that this is equivalent to saying that if $x_1 = 1$, then x_4, x_5 are **both** = 1, i.e., if $x_1>0 \Rightarrow x_4+x_5 \geq 2$, i.e, $2- x_4-x_5 \leq 0$

This gets us into the format of "$g_1(\mathbf{x})>0 \Rightarrow g_1(\mathbf{x})\leq 0$" to use the "trick" we saw in class, where $g_1(\mathbf{x})=x_1$ and $g_1(\mathbf{x})=2-x_4-x_5$.

Thus **both** $x_1>0$ and $2-x_4-x_5>0$ is impossible, i.e,

at least one of $x_1\leq 0$ and $2-x_4-x_5\leq 0$) **must** hold.

So we define y=0 or 1

Then (2.1) states that if y=0 then $x_1\leq 0$, i.e., $x_1=0$ (and we don't care about x_4, x_5…)

while (10) states that if y=1 then $2-x_4-x_5\leq 0$, i.e., $x_4+x_5\geq 2$, i.e., $x_4=x_5=1$

Alternatively, we could also express this constraint via

2. $x_4\geq x_1$, $x_5\geq x_1$ or **2.** $x_4+x_5\geq 2x_1$

Question 2 (*Q14, p. 503-504*)

Define y_j = 1 if disk j is stored, 0 otherwise; j = 1,2,…,10

Part (1)

Then the formulation is

Minimize $Z = 3y_1 + 5y_2 + y_3 + 2y_4 + y_5 + 4y_6 + 3y_7 + y_8 + 2y_9 + 2y_{10}$ (total storage)

st

$y_1 + y_2 + y_4 + y_5 + y_8 + y_9 \geq 1$ (file 1 must be stored)

$y_1 + y_3 \geq 1$ (file 2 must be stored)

$y_2 + y_5 + y_7 + y_{10} \geq 1$ (file 3 must be stored)

$y_3 + y_6 + y_8 \geq 1$ (file 4 must be stored)

$y_1 + y_2 + y_4 + y_6 + y_7 + y_9 + y_{10} \geq 1$ (file 5 must be stored)

All y_j = 0 or 1.

<u>Note</u>: This is an example of a set covering problem

Excel yielded the optimum solution $y_3=y_5=y_{10}=1$ and all other $y_j=0$, $Z^*=4$

Part (2)

Here we want that if either y_3 or y_5 (or both) are equal to 1, then y_2 MUST be 1
So we could add $y_2 \geq y_3$ and $y_2 \geq y_5$ to the problem constraints.

With the additional constraint the optimum solution has $y_1 = y_5 = y_8 = 1$ and all other $y_j = 0$, $Z^* = 5$

Question 3 *(Q43, p511)*
First note that the fixed and variable costs are as follows:

		Variable Costs ($ Billion) =500,000 cars * $/car		
Plant	Fixed Cost ($ Billion)	Taurus	Lincoln	Escort
1	7	6	8	9
2	6	7.5	9	5.5
3	4	8.5	9.5	6
4	2	9.5	11	7

Define $x_{ij}=1$ if Plant i (i=1,2,3,4) is used to produce type j (j=1,2,3) cars (Type 1 = Tauras, etc.); 0 otherwise. Also define $y_i=1$ if Plant i is used to produce any type of car; 0 otherwise (note that one plant will definitely not be used). Then expressing the objective in billions of dollars we have

Min $7y_1 + 6y_2 + 4y_3 + 2y_4 + 6x_{11}+8x_{12}+4.5x_{13}+7.5x_{21}+9x_{22}+5.5x_{23}+8.5x_{31}+9.5x_{32}+ 6x_{33}+9.5x_{41}+11x_{42}+7x_{43}$
st

A.

- $x_{11}+x_{12}+x_{13}\leq 1$ (Plant 1 produces no more than one car type)
- $x_{21}+x_{22}+x_{23}\leq 1$ (Plant 2 produces no more than one car type)
- $x_{31}+x_{32}+x_{33}\leq 1$ (Plant 3 produces no more than one car type)
- $x_{41}+x_{42}+x_{43}\leq 1$ (Plant 4 produces no more than one car type)

B.

- $x_{11}+x_{21}+x_{31}+x_{41}=1$ (Car type 1 is produced at exactly one plant)
- $x_{12}+x_{22}+x_{32}+x_{42}=1$ (Car type 2 is produced at exactly one plant)
- $x_{13}+x_{23}+x_{33}+x_{43}=1$ (Car type 3 is produced at exactly one plant)

C.

- $x_{11} \leq y_1$, $x_{12} \leq y_1$, $x_{13} \leq y_1$ (If Plant 1 is not used at all, then it produces nothing)
- $x_{21} \leq y_2$, $x_{22} \leq y_2$, $x_{23} \leq y_2$ (If Plant 2 is not used at all, then it produces nothing)
- $x_{31} \leq y_3$, $x_{32} \leq y_3$, $x_{33} \leq y_3$ (If Plant 3 is not used at all, then it produces nothing)
- $x_{41} \leq y_4$, $x_{42} \leq y_4$, $x_{43} \leq y_4$ (If Plant 4 is not used at all, then it produces nothing)

NOTE: You could also combine A and C via $x_{11}+x_{12}+x_{13}\leq y_1$, $x_{21}+x_{22}+x_{23}\leq y_2$ etc.

D.

- $y_3+y_4-1 \leq M(1-z)$
- $1-y_1 \leq Mz$

The last two constraint arises from "If 3 and 4 are used then 1 must be used." That is, "If $y_3+y_4\geq 2$, then $y_1\geq 1$,"

i.e., if $y_3+y_4>1$ then $y_1\geq 1$,
i.e., both $y_3+y_4>1$ **and** $y_1<1$ is not possible,
i.e., at least one of $y_3+y_4-1\leq 0$ or $1-y_1\leq 0$ must hold.

This yields the last two constraints.

- Alternatively, we could also express this last condition via $y_1 \geq y_3 + y_4 - 1$ (so if both y_3 & y_4 are = 1, then the RHS is = 1 so that y_1 is also = 1, if not the RHS is = 0 or -1, so that y_1 could be 0 or 1)

All variables $\in(0,1)$

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Alternative (and somewhat more complex…) Formulation:
We could also dispense with the y_i variables altogether by pulling in the fixed cost for each plant in each year into the operating costs and altering the last two sets of constraints as follows:

Min $13x_{11}+15x_{12}+11.5x_{13}+13.5x_{21}+15x_{22}+11.5x_{23}+12.5x_{31}+13.5x_{32}+10x_{33}+11.5x_{41}+13x_{42}+9x_{43}$
st
Same constraints as in **A.** and **B.** above, and replace **C.** and **D.** with

- $x_{31}+x_{32}+x_{33}+x_{41}+x_{42}+x_{43}-1 \leq Mz$
- $x_{11}+x_{12}+x_{13}-1 \geq -M(1-z)$

All variables = 0 or 1.

Note that **A.** and **B.** ensure that exactly 3 of the four plants are used to produce 3 different cars; so the fixed cost will definitely be counted in the objective only once for each of the 3 plants that are used.

The last set of constraints simply ensure that **if one of (x_{31}, x_{32}, x_{33}) and one of (x_{41}, x_{42}, x_{43}) are equal to 1** (note that the constraints in A preclude more than one in each set from being equal to 1….), **then one of (x_{11}, x_{12}, x_{13}) is also equal to 1**.
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The optimum solution has $y_1=y_3=y_4=1$; $x_{11}=x_{32}=x_{43}=1$ with an objective function value = 35.5

Question 4

Define y_j = 1 if location j is selected and 0 otherwise, j=1,2,...,M
and x_{ij} = 1 if district i is assigned to the firehouse at location j, i=1,2,...,N, j=1,2,...,M

The constraints are as follows:

1) $$\sum_{j=1}^{M} x_{ij} = 1, \quad i = 1,2,\ldots,N$$ (every district is assigned to exactly one firehouse)

2) $$\sum_{i=1}^{N} x_{ij} \leq y_j N, \quad j = 1,2,\ldots,M$$ (no district is assigned to an unused location)

(note that the sum of the x_{ij} over all i can obviously never exceed N)

3) $\sum_{i=1}^{N} p_i x_{ij} = s_j, \quad j = 1,2,\ldots,M$ (total population served by location j)

4) $\sum_{j=1}^{M} f_j(s_j) \le B$ (budgetary constraint)

(Note that 3 and 4 could be combined and s_j eliminated...)

5) $y_1 + y_2 \ge 2z_1$

6) $y_3 + y_4 \ge 2z_2$

7) z_1+z_2=1 (or ≥1)

(5, 6 and 7 together ensure that at least one of $y_1+y_2 \ge 2$ and $y_3+y_4 \ge 2$ is satisfied)

Suppose we define d_i as the distance to district i from its assigned firehouse and $D=\text{Max}_i(d_i)$

8) $d_i = \sum_{j=1}^{M} d_{ij} x_{ij}$ (CASE 1)

8') $D \ge \sum_{j=1}^{M} d_{ij} x_{ij}, \quad i = 1,2,\ldots,N$ (CASE 2)

9) x_{ij}, y_j, z_1, z_2 all 0 or 1 for i=1,2,...,N, j=1,2,...,M.

The objective is to (case 1) **Minimize (Σd_i)/N**, or (case 2) **Minimize D**