

Fundamental Theorem of Linear Programming

THEOREM: For a feasible linear program in its standard form, the optimum value of the objective over its nonempty feasible region is (a) either unbounded or (b) is achievable at least at one extreme point of the feasible region.

We will see what is meant by “standard form” very shortly...

More generally, the above theorem and the graphs we saw tell us that every LP is in exactly one of the following states:

1. Feasible with a **unique optimum** solution - clause (b) of the fundamental theorem
2. Feasible with **infinitely many optima** - clause (b) of the fundamental theorem
3. Feasible, with **no** optimum solution because the objective is **unbounded** - clause (a) of the fundamental theorem
4. **Infeasible**, and therefore, with **no** optimum solution

The Simplex Method - Development

Assume for now that we have a feasible LP and that the objective is bounded (i.e., we are in States 1 or 2). Then the following are true:

- A. The LP has at least one optimal corner point (or *extreme point*)
 - 1. If in State 1, exactly one extreme point is optimal
 - 2. If in State 2, at least two **adjacent** (neighboring) extreme points are optimal
- B. The number of extreme points is finite
- C. *If the objective function at some extreme point is as good as, or better than it is at all of its adjacent extreme points, then this extreme point is optimal for the LP*

This suggests a simple algorithmic approach to finding the optimum of a feasible and bounded LP...

The Simplex Algorithm – An Outline

STEP 0 (Initialization): Find an initial extreme point and make it the current candidate (if one cannot be found the LP is in state 4, i.e. it is infeasible – so STOP).

STEP 1 (Stopping Criterion Check): Is the objective at the current extreme point *at least as good or better* than it is at all of its adjacent (neighboring) extreme points? If so this must be the optimal extreme point (via **C**) – so STOP. If not, go to Step 2.

STEP 2 (Iterative Step): One (or more accurately, at least one...) of the adjacent extreme points is better – so make it the current candidate. Then return to Step 1.

QUESTIONS:

- 1) How to find an initial extreme point?
- 2) What is the **algebraic** characterization of an “extreme point?”
- 3) What is the algebraic characterization of **adjacent** extreme points, i.e., how to move from an extreme point to one of its neighbors (in Step 2)?

Standard Form of a Linear Program

Three conditions must be met:

1) All constraints must be stated as **equalities** of the form

$$g_k(\mathbf{x})=b_k, \text{ where } g_k(\mathbf{x}) \text{ is a linear function of } \mathbf{x}.$$

2) The RHS for each constraint must be nonnegative, i.e., all

$$b_k \geq 0$$

3) All variables must be nonnegative, i.e., $x_i \geq 0$ for all i .

Thus the LP (with m constraints and n variables) looks like

$$\text{Min (or Max) } c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{st } a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

:

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_1, x_2, \dots, x_n \geq 0.$$

where $b_1, b_2, \dots, b_m \geq 0$.

More compactly, let $\mathbf{c}=[c_1 \ c_2 \ \dots \ c_n]$ and

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then the LP may be restated as

$$\begin{aligned} &\text{Min (or Max) } \mathbf{cx}, \\ &\text{st } \mathbf{Ax}=\mathbf{b}, \mathbf{x}\geq\mathbf{0}, \text{ where } \mathbf{b}\geq\mathbf{0}. \end{aligned}$$

This final system of constraint equations of the LP in standard form is called the *AUGMENTED SYSTEM*.

Consider such an augmented system of n variables in m equations, where $m \leq n$. Note that the augmented system **does not include** the nonnegativity conditions!

Also note that this system has three possible states: (1) no solution, (2) exactly one solution, or (3) infinitely many solutions.

Feasible solution: Any solution to the augmented system that also satisfies nonnegativity is called a **feasible solution**

E.g. Consider the following constraints and the corresponding augmented system:

$$\begin{aligned} x_1 + 2x_2 &\leq 120 \\ x_1 + x_2 &\leq 90 \\ x_1 &\leq 70 \\ x_2 &\leq 50 \end{aligned}$$

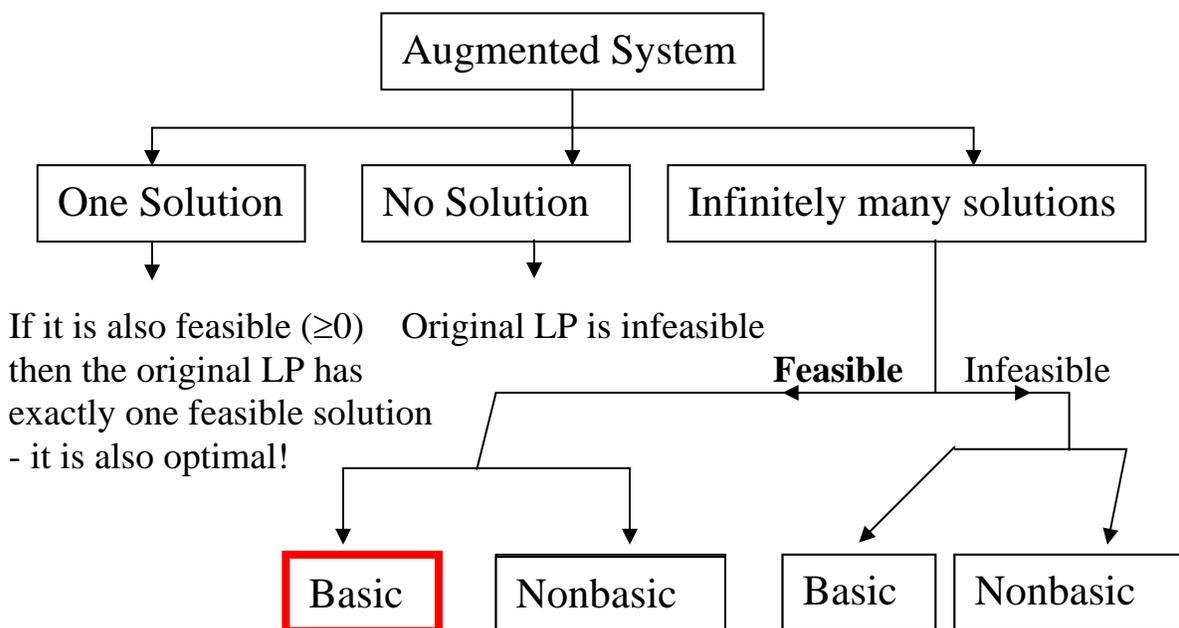
$x_1 + 2x_2 + S_1$	$= 120$
$x_1 + x_2 + S_2$	$= 90$
$x_1 + S_3$	$= 70$
$x_2 + S_4$	$= 50$

Note that $n=6$ and $m=4$.

- 1) An (infeasible) solution is $[x_1=70, x_2=50, S_1=-50, S_2=-30, S_3=0, S_4=0]$
- 2) A **feasible** solution is $[x_1=20, x_2=20, S_1=60, S_2=50, S_3=50, S_4=30]$

Basic Solution: Suppose we **fix** $(n-m)$ out of the n variables at zero, and try to solve the system of m equations in the remaining m variables. If a solution to this exists, then it is called a **basic solution**. Solution (1) on the previous page is an example of a basic solution.

Basic Feasible Solution: A basic solution that also satisfies nonnegativity is called a **basic feasible solution (BFS)**. An example of a BFS is $[x_1=20, x_2=50, S_1=0, S_2=20, S_3=50, S_4=0]$. Note that (1) is **not** a BFS even though it **is** a basic solution!



FACT: Each **BFS** corresponds to an extreme point of the feasible region.

In a basic solution, the $(n-m)$ variables that are chosen to be fixed at zero are called the **nonbasic** variables and the remaining m variables are called **basic** variables.

Note that we can choose $(n-m)$ out of the n variables in

$\binom{n}{n-m} = \binom{n}{m} = \frac{n!}{(n-m)!m!}$ different ways. Thus we can get a

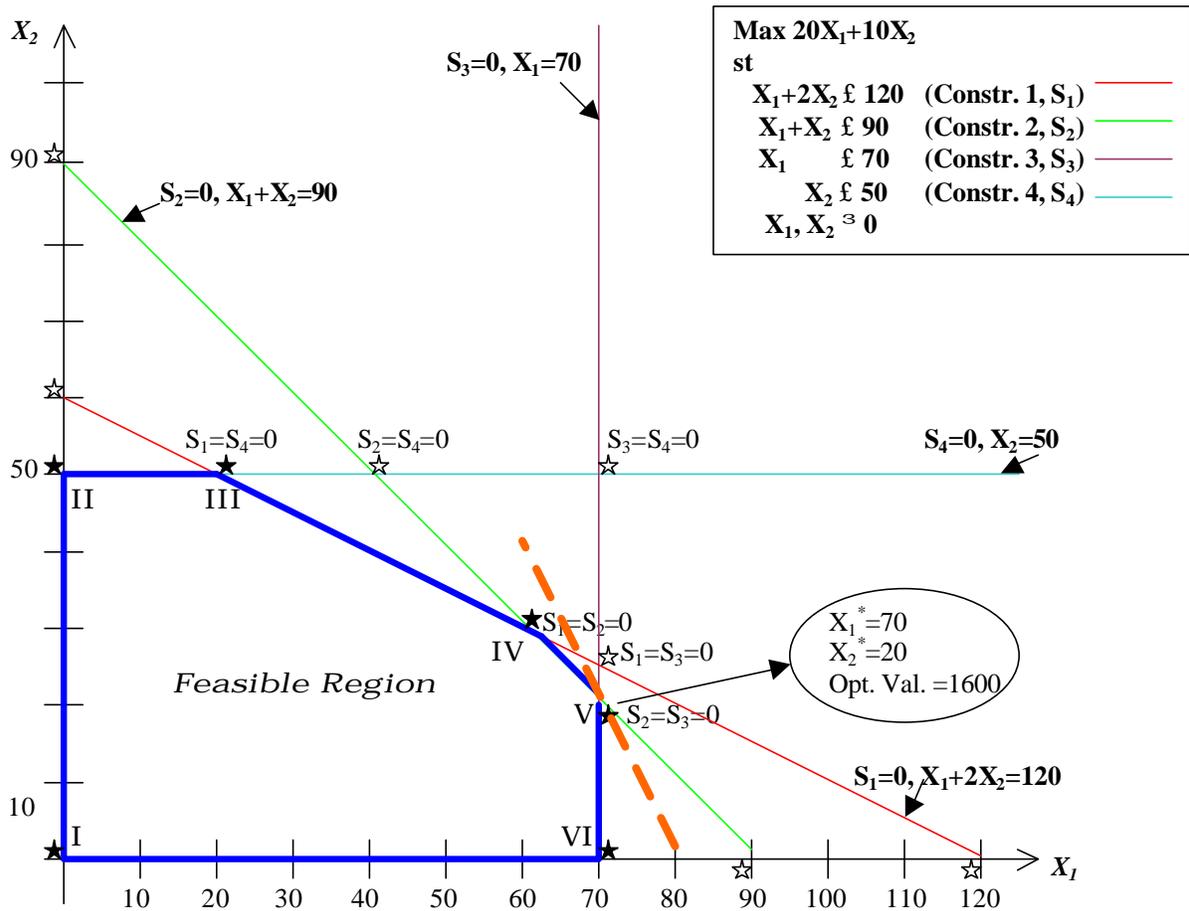
maximum of this many different basic solutions (some of which will also be basic feasible solutions) – in practice all of these may not exist since in some cases, the resulting $m \times m$ system may not have a solution!

In our example, $n=6$ and $m=4$ so that $\binom{6}{6-4} = \frac{6!}{(6-4)!4!} = 15$

The 15 combinations of nonbasic variables are:

- 1) x_1, x_2
- 2) x_1, S_1 6) x_2, S_1
- 3) x_1, S_2 7) x_2, S_2 10) S_1, S_2
- 4) x_1, S_3 8) x_2, S_3 11) S_1, S_3 13) S_2, S_2
- 5) x_1, S_4 9) x_2, S_4 12) S_1, S_4 14) S_2, S_4 15) S_3, S_4

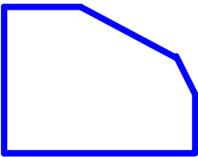
Try to locate each of these on the graphical representation!



1. \star, \star : BASIC SOLUTIONS (13 of them). Note that there should be $\binom{6}{4} = 15$ of these, but only 13 exist because $X_1 = 70$ is parallel to the X_2 -axis and $X_2 = 50$ is parallel to the X_1 -axis.

2. \star : BASIC FEASIBLE SOLUTION: (6 of the above 13, numbered I, II, III, IV, V, VI)

3. $- - -$: Contour of the objective function corresponding to a value of 1600

4.  : Set of all Feasible solutions to the LP