GETTING AN INITIAL *BFS* WHEN THE ORIGIN IS NOT IN THE FEASIBLE REGION

This could happen in the presence of $\geq$ and = constraints… e.g.,

Max $z = 2x_1 + 5x_2 + 3x_3$

st $3x_1 - 6x_2 \geq 30$

$6x_1 + 12x_2 + 3x_3 = 75$; $x_1, x_2, x_3 \geq 0$.

In standard form this yields

Max $z = 2x_1 + 5x_2 + 3x_3$

st $3x_1 - 6x_2 - S_1 = 30$

$6x_1 + 12x_2 + 3x_3 = 75$; $x_1, x_2, x_3, S_1 \geq 0$.

An attempt at an initial tableau:

Row	Basic	Z	x_1	x_2	x_3	S_1	RHS
0	Z	1	-2	-5	-3	0	0
1	***??***	0	3	-6	0	-1	30
2	***??***	0	6	12	3	0	75

NOT READY – NOT IN CANONICAL FORM!!

Notice that this is not in the "usual" form we have seen thus far.
Let us add "artificial" variables A_1 and A_2 to the constraints. This yields…

$$\text{Max } z = 2x_1 + 5x_2 + 3x_3 + 0S_1 - MA_1 - MA_2$$

$$\begin{aligned} \text{st} \quad & 3x_1 - 6x_2 \quad -S_1 + A_1 = 30 \\ & 6x_1 + 12x_2 + 3x_3 \quad + A_2 = 75 \\ & x_1, x_2, x_3, S_1, A_1, A_2 \geq 0. \end{aligned}$$

The tableau now look as follows:

Row	Basic	Z	x_1	x_2	x_3	S_1	A_1	A_2	RHS
0	Z	1	-2	-5	-3	0	M	M	0
1	A_1	0	3	-6	0	-1	1	0	30
2	A_2	0	6	12	3	0	0	1	75

Still not ***quite*** in the canonical form! Let $Eq\ 0 \leftarrow Eq\ 0 + (-M)(Eq\ 1) + (-M)(Eq\ 2)$

This yields

Row	Basic	Z	x_1	x_2	x_3	S_1	A_1	A_2	RHS
0	Z	1	-9M-2	-6M-5	-3M-3	M	0	0	-105M
1	A_1	0	3	-6	0	-1	1	0	30
2	A_2	0	6	12	3	0	0	1	75

Finally in correct canonical form!

An example with ≥ and = constraints…

Max $z = 12x_1 + 20x_2 + 8x_3$

st
$$
\begin{aligned}
3x_1 \qquad\quad -2x_3 &\geq -2\\
-4x_1 \; -x_2 + 12x_3 &\geq 4\\
-x_1 \qquad\quad + 3x_3 &\leq -6\\
3x_1 + 4x_2 - 6x_3 &= -12\\
x_2 + 9x_3 &= 31
\end{aligned}
\qquad\qquad x_1, x_2, x_3 \geq 0.
$$

First, convert to standard form, and then to canonical form

- an "isolated" (basic) variable for each equation that has a coefficient of +1

Equation 1 $3x_1 - 2x_3 \geq -2 \Rightarrow 3x_1 - 2x_3 - S_1 = -2 \Rightarrow \boxed{-3x_1 + 2x_3 + S_1 = 2}$

Equation 2 $-4x_1 - x_2 + 12x_3 \geq 4 \Rightarrow -4x_1 - x_2 + 12x_3 - S_2 = 4 \Rightarrow \boxed{-4x_1 - x_2 + 12x_3 - S_2 + A_2 = 4}$

Equation 3 $-x_1 + 3x_3 \leq -6 \Rightarrow -x_1 + 3x_3 + S_3 = -6 \Rightarrow x_1 - 3x_3 - S_3 = 6 \Rightarrow \boxed{x_1 - 3x_3 - S_3 + A_3 = 6}$

Equation 4 $3x_1 + 4x_2 - 6x_3 = -12 \Rightarrow -3x_1 - 4x_2 + 6x_3 = 12 \Rightarrow \boxed{-3x_1 - 4x_2 + 6x_3 + A_4 = 12}$

Equation 5 $x_2 + 9x_3 = 31 \Rightarrow \boxed{x_2 + 9x_3 + A_5 = 31}$

Final LP in Standard Form

$$
\begin{array}{lrrrrrrrrrrl}
\text{Max } z = & 12x_1 & + 20x_2 & + 8x_3 & & & & - MA_2 & - MA_3 & - MA_4 & - MA_5 & \\
\text{st} & -3x_1 & & + 2x_3 & + S_1 & & & & & & & = 2 \\
 & -4x_1 & - x_2 & + 12x_3 & & - S_2 & & + A_2 & & & & = 4 \\
 & x_1 & & - 3x_3 & & & - S_3 & & + A_3 & & & = 6 \\
 & -3x_1 & - 4x_2 & + 6x_3 & & & & & & + A_4 & & = 12 \\
 & & x_2 & + 9x_3 & & & & & & & + A_5 & = 31
\end{array}
$$

all variables ≥ 0.

Final Tableau

Basic	Z	x_1	x_2	x_3	S_1	S_2	A_2	S_3	A_3	A_4	A_5	RHS
z	1	-12	-20	-8	0	0	M	0	M	M	M	0
S_1	0	-3	0	2	1	0	0	0	0	0	0	2
A_2	0	-4	-1	12	0	-1	1	0	0	0	0	4
A_3	0	1	0	-3	0	0	0	-1	1	0	0	6
A_4	0	-3	-4	6	0	0	0	0	0	1	0	12
A_5	0	0	1	9	0	0	0	0	0	0	1	31

Eq. 0← Eq. 0 - M Eq.2 - M Eq.3 - M Eq.4 - M Eq.5

Z	1	(6M -12)	(4M -20)	(-24M -8)	0	M	0	M	0	0	0	-54M
S_1	0	-3	0	2	1	0	0	0	0	0	0	2
A_2	0	-4	-1	12	0	-1	1	0	0	0	0	4
A_3	0	1	0	-3	0	0	0	-1	1	0	0	6
A_4	0	-3	-4	6	0	0	0	0	0	1	0	12
A_5	0	0	1	9	0	0	0	0	0	0	1	31

Example: Max $z = 5000x_1 + 4000x_2$ $-MA_3$

st $10x_1 + 15x_2 \leq 150 \Rightarrow 10x_1 + 15x_2 + S_1 = 150$

$20x_1 + 10x_2 \leq 160 \Rightarrow 20x_1 + 10x_2 + S_2 = 160$

$30x_1 + 10x_2 \geq 135 \Rightarrow 30x_1 + 10x_2 - S_3 + A_3 = 150$; everything$\geq 0$

Basic	Z	x_1	x_2	S_1	S_2	S_3	A_3	RHS
Z	1	-5000	-4000	0	0	0	M	0
S_1	0	10	15	1	0	0	0	150
S_2	0	20	10	0	1	0	0	160
A_3	0	30	10	0	0	-1	1	135

Not yet in canonical form…

Eq. 0 $\leftarrow$ Eq. 0 + (-M)(Eq. 3)

Basic	Z	x_1	x_2	S_1	S_2	S_3	A_3	RHS	
Z	1	-30M-5000	-10M-4000	0	0	M	0	-135M	
S_1	0	10	15	1	0	0	0	150	150/10=15
S_2	0	20	10	0	1	0	0	160	160/20=8
→ A_3	0	(30)	10	0	0	-1	1	135	135/30=4.5

Finally, in canonical form!

ITERATION 1

Basic	Z	x_1	x_2	S_1	S_2	S_3	A_3	RHS	
Z	1	0	-7000/3	0	0	-500/3	M+500/3	22,500	
S_1	0	0	**35/3**	1	0	1/3	-1/3	105	105/(35/3)=9
S_2	0	0	10/3	0	1	2/3	-2/3	70	70/(10/3)=21
x_1	0	1	1/3	0	0	-1/30	1/30	4.5	4.5/(1/3)=13.5

ITERATION 2

Basic	Z	x_1	X_2	S_1	S_2	S_3	A_3	RHS	
Z	1	0	0	200	0	-100	M+100	43,500	
x_2	0	0	1	3/35	0	1/35	-1/35	9	9/(1/35)=315
S_2	0	0	0	-10/35	1	**4/7**	-4/7	40	40/(4/7)=70
x_1	0	1	0	-1/35	0	-3/70	3/70	1.5	∞

ITERATION 3

Basic	Z	x_1	X_2	S_1	S_2	S_3	A_3	RHS
Z	1	0	0	150	175	0	M	50,500
x_2	0	0	**1**	1/10	-1/20	0	0	7
S_3	0	0	0	-1/2	7/4	1	-1	70
x_1	0	1	0	-1/20	3/40	0	0	4.5

$x^*=[4.5 \quad 7]^T$; $Z^* = 50{,}500$

Example 2:

Max $z = 2x_1 + 5x_2 + 3x_3 + 0S_1 - MA_1 - MA_2$

st $3x_1 - 6x_2 - S_1 + A_1 = 30$

$6x_1 + 12x_2 + 3x_3 + A_2 = 75$

$x_1, x_2, x_3, S_1, A_1, A_2 \geq 0.$

Basic	Z	x_1 ↓	x_2	x_3	S_1	A_1	A_2	RHS	
Z	1	-9M-2	-6M-5	-3M-3	M	0	0	-105M	
→ A_1	0	(3)	-6	0	-1	1	0	30	30/3=10
A_2	0	6	12	3	0	0	1	75	75/6=12.5

Basic	Z	x_1	x_2 ↓	x_3	S_1	A_1	A_2	RHS
Z	1	0	-24M-9	-3M-3	-2M-2/3	3M+2/3	0	-15M+20
x_1	0	1	-2	0	-1/3	1/3	0	10
A_2	0	0	(24)	3	2	-2	1	15

Basic	Z	x_1	x_2	x_3	S_1	A_1	A_2	RHS	
Z	1	0	0	-15/8	1/12	M-1/12	M+3/8	205/8	
x_1	0	1	0	1/4	-1/6	1/6	1/12	45/4	(45/4)/(1/4)=45
x_2	0	0	1	1/8	1/12	-1/12	1/24	5/8	(5/8)/(1/8)=5

Basic	Z	X_1	x_2	x_3	S_1	A_1	A_2	RHS
Z	1	0	15	0	4/3	M-4/3	M+1	35
x_1	0	1	-2	0	-1/3	1/3	0	10
x_3	0	0	8	1	2/3	-2/3	1/3	5

Optimum Solution: $x^*=[10 \quad 0 \quad 5]^T$; $z^*=35$

The Two-Phase Method

A (more sophisticated) alternative to the Big-M method

- The constraints are set up as in the Big-M method, but in Phase-I we **Minimize the sum of the artificial variables**
  - Phase 1 Objective: Min $W = \sum_i A_i$
- At the optimal iteration for this problem (end of Phase 1):
  - If the optimal Phase 1 objective is positive (i.e., there are artificial variables in the basis), then the original problem is infeasible
  - If all artificial variables are nonbasic (so that the Phase 1 optimum = 0) we have a legitimate basis. Continue to Phase 2: **Replace the Phase 1 objective with the original objective and continue as usual**.
  - If the objective is zero but there are artificial variables in the basis **at the zero level**, then drop all nonbasic artificial variables, and continue to Phase 2, but ensure that any basic artificial variable that is zero never becomes positive in Phase 2 (this case is rare; we will ignore it…)

Example: Max $z = 5000x_1 + 4000x_2$

st $10x_1 + 15x_2 \leq 150 \Rightarrow 10x_1 + 15x_2 + S_1 = 150$

$20x_1 + 10x_2 \leq 160 \Rightarrow 20x_1 + 10x_2 + S_2 = 160$

$30x_1 + 10x_2 \geq 135 \Rightarrow 30x_1 + 10x_2 - S_3 + A_3 = 150$; everything ≥ 0

PHASE 1: Min $W = A_3$

Basic	W	x_1	x_2	S_1	S_2	S_3	A_3	RHS
W	1	0	0	0	0	0	-1	0
S_1	0	10	15	1	0	0	0	150
S_2	0	20	10	0	1	0	0	160
A_3	0	30	10	0	0	-1	1	135

To put this in canonical form, do Row 0 $\leftarrow$ Row 0 +Row 3

Basic	W	x_1	x_2	S_1	S_2	S_3	A_3	RHS	
W	1	30	10	0	0	-1	0	135	
S_1	0	10	15	1	0	0	0	150	150/10=15
S_2	0	20	10	0	1	0	0	160	160/20=8
A_3	0	(30)	10	0	0	-1	1	135	135/30=4.5

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Enter x_1 and remove A_3

Basic	W	x_1	x_2	S_1	S_2	S_3	A_3	RHS
W	1	0	0	0	0	0	-1	0
S_1	0	0	35/3	1	0	1/3	-1/3	105
S_2	0	0	10/3	0	1	2/3	-2/3	70
x_1	0	1	1/3	0	0	-1/3	1/3	4.5

End of Phase 1. Now introduce original objective (to be **maximized**)

Basic	Z	X_1	x_2	S_1	S_2	S_3	RHS
Z	1	-5000	-4000	0	0	0	0
S_1	0	0	35/3	1	0	1/3	105
S_2	0	0	10/3	0	1	2/3	70
x_1	0	1	1/3	0	0	-1/3	4.5

Put into canonical form: Row 0←Row 0+5000*Row 3.
SAME TABLEAU AS IN ITERATION 1 OF THE Big-M method. Now continue as usual…

	Basic	Z	X_1	x_2 ↓	S_1	S_2	S_3	RHS	
	Z	1	0	-7000/3	0	0	-5000/3	22,500	
→	S_1	0	0	(35/3)	1	0	1/3	105	105/(35/3)=9
	S_2	0	0	10/3	0	1	2/3	70	70/(10/3)=21
	x_1	0	1	1/3	0	0	-1/3	4.5	40.5 (1/3)=13.5

Example 2:

Max $z = 2x_1 + 5x_2 + 3x_3$

st $3x_1 - 6x_2 - S_1 + A_1 = 30$

$6x_1 + 12x_2 + 3x_3 + A_2 = 75$

$x_1, x_2, x_3, S_1, A_1, A_2 \geq 0.$

<u>PHASE 1</u> (Minimize A_1+A_2)

Basic	W	x_1	x_2	x_3	S_1	A_1	A_2	RHS
W	1	0	0	0	0	-1	-1	0
A_1	0	3	-6	0	-1	1	0	30
A_2	0	6	12	3	0	0	1	75

Canonical Form

Basic	W	x_1	x_2	x_3	S_1	A_1	A_2	RHS	
W	1	9	6	3	-1	0	0	105	
→ A_1	0	(3)	-6	0	-1	1	0	30	30/3=10
A_2	0	6	12	3	0	0	1	75	75/6=12.5

Basic	W	x_1	x_2	x_3	S_1	A_1	A_2	RHS
W	1	0	24	3	2	-3	0	15
x_1	0	1	-2	0	-1/3	1/3	0	10
A_2	0	0	24	3	2	-2	1	15

Basic	W	x_1	x_2	x_3	S_1	A_1	A_2	RHS
W	1	0	0	0	0			0
x_1	0	1	0	1/4	-1/6			45/4
x_2	0	0	1	1/8	1/12			5/8

PHASE 2

Basic	Z	x_1	x_2	x_3	S_1	RHS
Z	1	-2	-5	-3	0	0
x_1	0	1	0	1/4	-1/6	45/4
x_2	0	0	1	1/8	1/12	5/8

Basic	Z	x_1	x_2	x_3	S_1	RHS
Z	1	0	0	-15/8	1/12	205/8
x_1	0	1	0	1/4	-1/6	45/4
x_2	0	0	1	1/8	1/12	5/8

Same starting table as before!