POSTOPTIMALITY ANALYSIS (*aka* SENSITIVITY ANALYSIS)

Objective: To analyze the optimum solution to see how sensitive this solution is w.r.t. the *cost coefficients* and the *right-hand-side* values.

Consider our example:

$$\text{Maximize } z = 5000x_1 + 4000x_2$$

st

$$10x_1 + 15x_2 \leq 150$$
$$20x_1 + 10x_2 \leq 160$$
$$30x_1 + 10x_2 \geq 135, \quad \text{all } x_i \geq 0.$$

The optimum solution was $x_1^*=4.5$, $x_2^*=7$, $z^*=50{,}500$

LINDO Output

LP OPTIMUM FOUND AT STEP 2

OBJECTIVE FUNCTION VALUE

Z) 50500.00

VARIABLE	VALUE	REDUCED COST
X1	4.500000	0.000000
X2	7.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
1)	0.000000	150.000000
2)	0.000000	175.000000
3)	70.000000	0.000000

NO. ITERATIONS= 2

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	5000.000000	3000.000000	2333.333252
X2	4000.000000	3500.000000	1500.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
1	150.000000	90.000000	70.000000
2	160.000000	140.000000	40.000000
3	135.000000	70.000000	INFINITY

MS-Excel Solver Output

Microsoft Excel 12.0 Answer Report
Worksheet: [SUBSTITUTION OOS.xls]Sheet4
Report Created: 10/15/2008 4:35:07 PM

Target Cell (Max)

Cell	Name	Original Value	
A4	Z	50500	50500

Adjustable Cells

Cell	Name	Original Value	Final Value
A1	X_1	4.5	4.5
A2	X_2	7	7

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
A6	Constraint1	150	A6<=150	Binding	0
A7	Constraint2	160	A7<=160	Binding	0
A8	Constraint3	205	A8>=135	Not Binding	70

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [SUBSTITUTION OOS.xls]Sheet4
Report Created: 10/15/2008 4:35:07 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
A1	X_1	4.5		5000	3000	2333.333333
A2	X_2	7		4000	3500	1500

Constraints

Cell	Name	Value	Price	R.H. Side	Allowable Increase	Allowable Decrease
A6	Constraint1	150	150	150	90	70
A7	Constraint2	160	175	160	140	40
A8	Constraint3	205	0	135	70	1E+30

A. Sensitivity to Cost Coefficients

Suppose we wish to examine variations in c_1 (the coefficient for x_1) from its current value of 5000: say a variation of Δc_1 units.

Let us say the actual value is $c_1' = c_1 + \Delta c_1$ (=5000+Δc_1, in this case…).

We can write

$z = c_1' x_1 + 4000 x_2 \;\approx\; x_2 = (-c_1'/4000) * x_1 + (z/4000)$

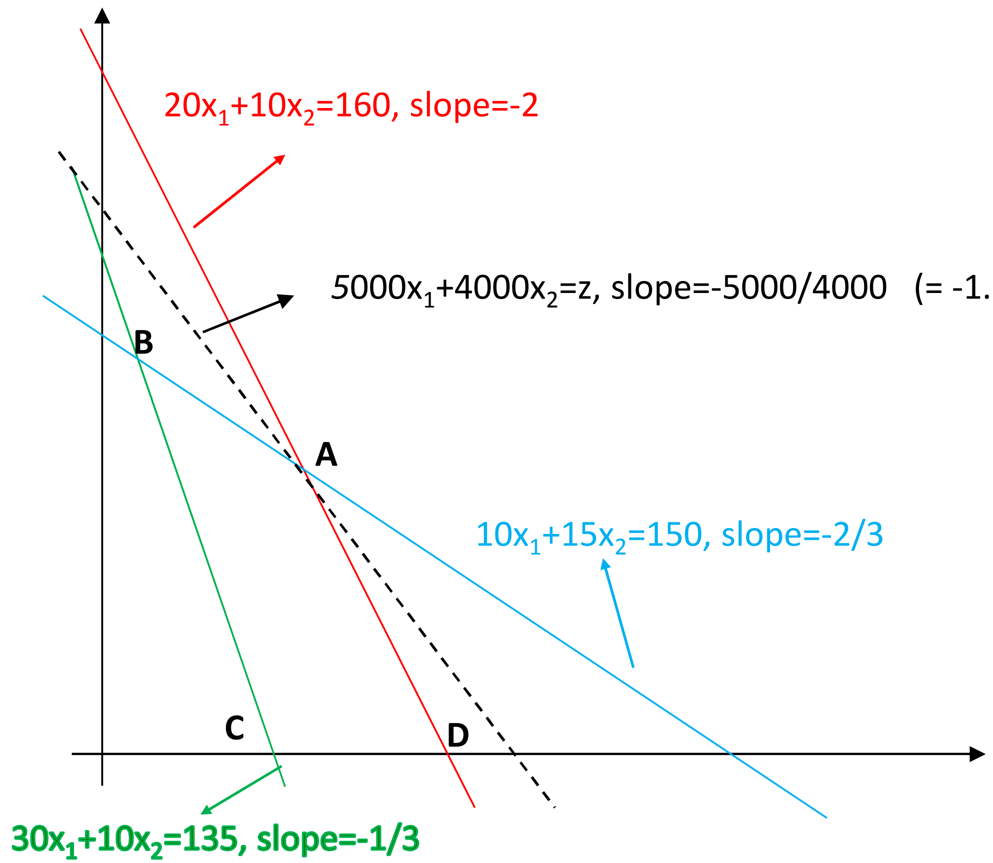
The slope of the above line is ($-c_1'/4000$); currently this slope has a value of -5000/4000 = -1.25

If c_1' increases the slope becomes more negative, and conversely, if c_1' decreases the slope becomes less negative.

In other word, the isocost line representing the objective rotates in a clockwise (≈ more negative) or a counter-clockwise (≈ less negative) direction.

Sensitivity to Cost Coefficients (cont'd)

If this rotation is small the optimum solution might be unchanged, but for a sufficiently large tilt the optimum could shift to a neighboring corner point.



Sensitivity to Cost Coefficients (cont'd)

In the above picture, the current value of c_1=5000 so that the current slope is -(5000/4000) = -1.25.

The extreme point that is currently optimal (**A**) is unchanged as long as the **new** slope (if c_1 is actually equal to c_1') lies between -2 and -2/3:

$$-2 \leq (-c_1'/4000) \leq -2/3 \Rightarrow -8000 \leq -c_1' \leq -8000/3 \Rightarrow$$

$$8000/3 \leq c_1' \leq 8000, \text{ i.e., } \boxed{2667 \leq c_1' \leq 8000}$$

So, for the basis to not change (i.e., for the optimum solution to remain at point **A**), the max allowable increase is 8000-5000=3000, and the max allowable decrease is 5000 - 2667 = 2333.

$$\text{i.e., } \boxed{\mathbf{-2333 \leq \Delta c_1 \leq 3000}}.$$

When the increase exceeds 3000 the optimum shifts to D, and when the decrease exceeds 2333 the optimum shifts to B.

A. Sensitivity to Right-Hand-Side Values

Now suppose we wish to examine variations in b_1 (the RHS for constraint 1) from its current value of 150.

Let us say the actual value is $b_1' = b_1 + \Delta b_1$ (=150+Δb_1, in this case…).

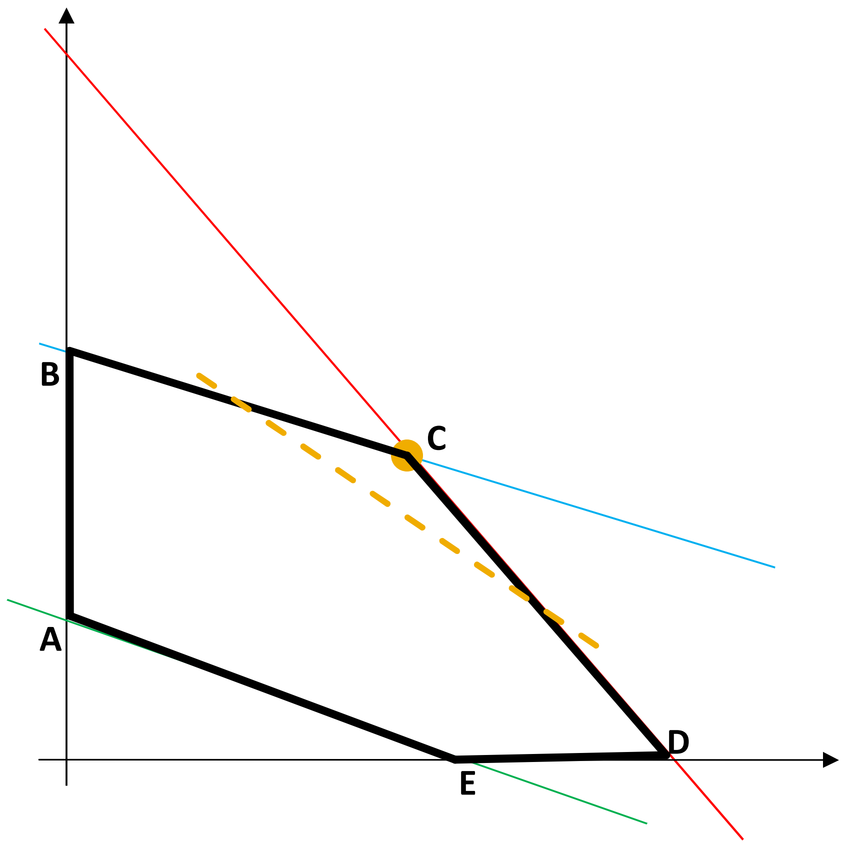
Consider the line $10x_1 + 15x_2 = b_1'$

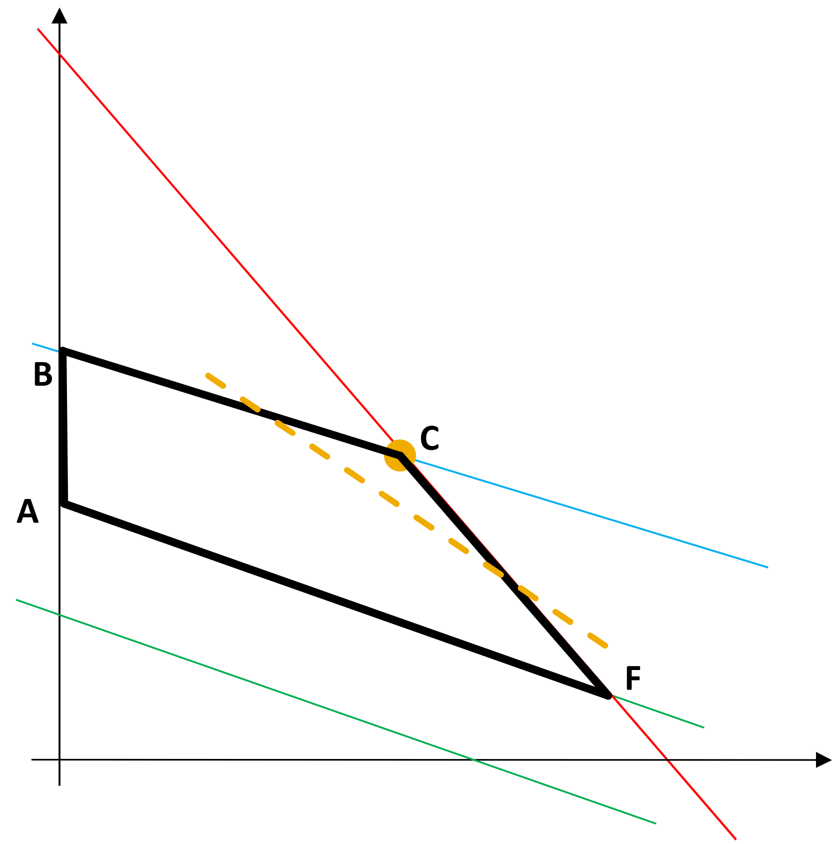
As the value of b_1' changes, the slope is unchanged but the line moves parallel to itself - either "upward" (if it increases in value) or "downward" (if it decreases in value).

In general, for the case where the RHS value changes, one of several different things could happen:

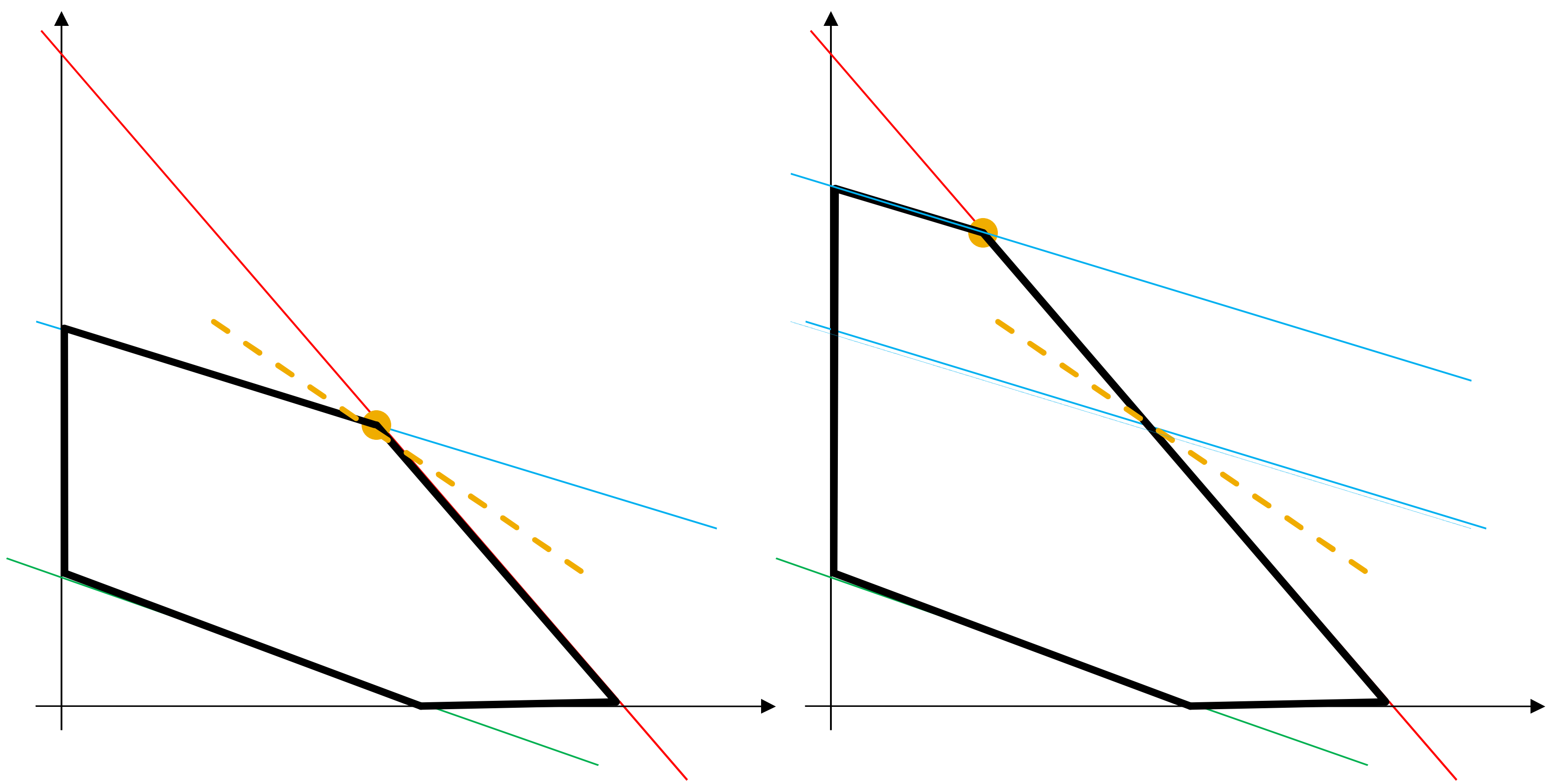
1. The current optimum point might be unaffected and remain optimum.
2. The current optimum point might not be an extreme point any longer and thus we would have a new optimum (extreme) point, which has
    a. either the same set of variables being basic
    b. or a different set of variables being basic
3. The entire problem might become infeasible (for a sufficiently large increase or decrease)

1. Optimum is unaffected

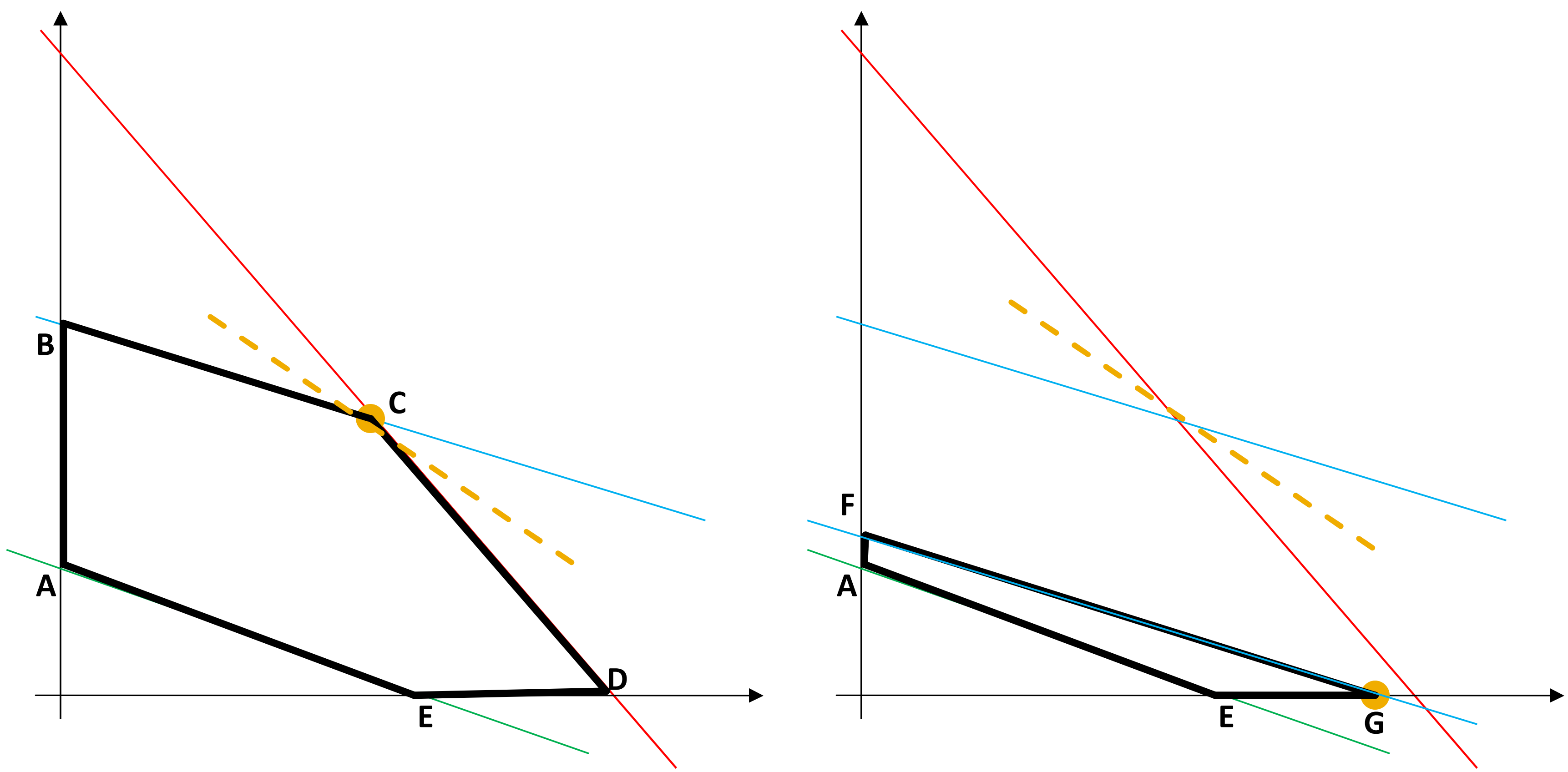




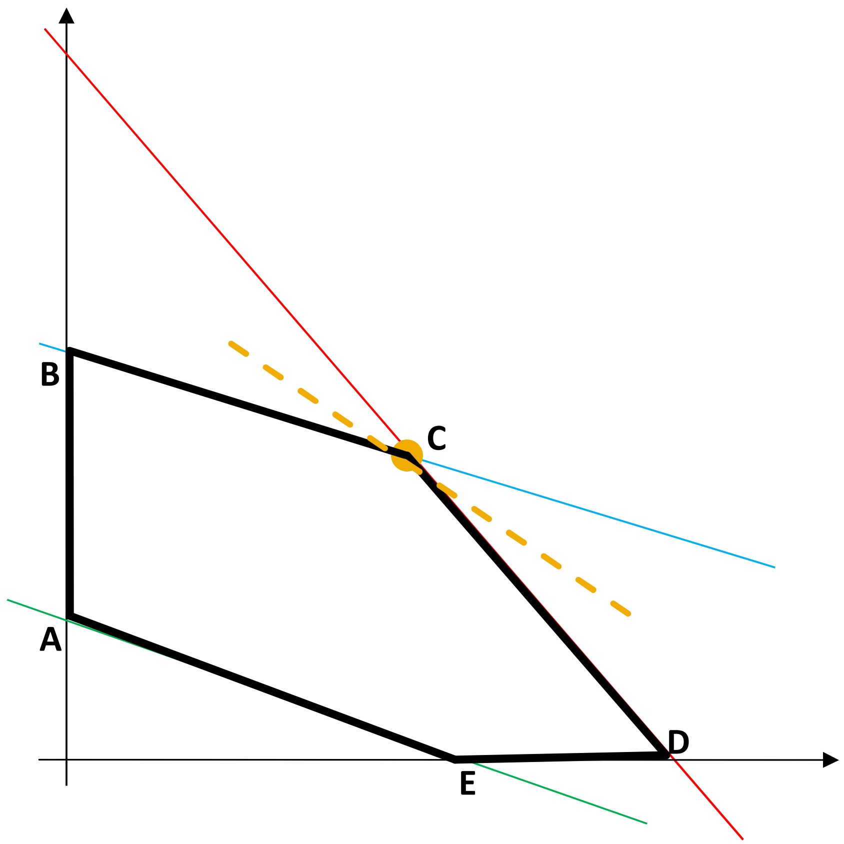
2a. New optimum with same basic variables: previous optimum is not an extreme point (BFS) any more…

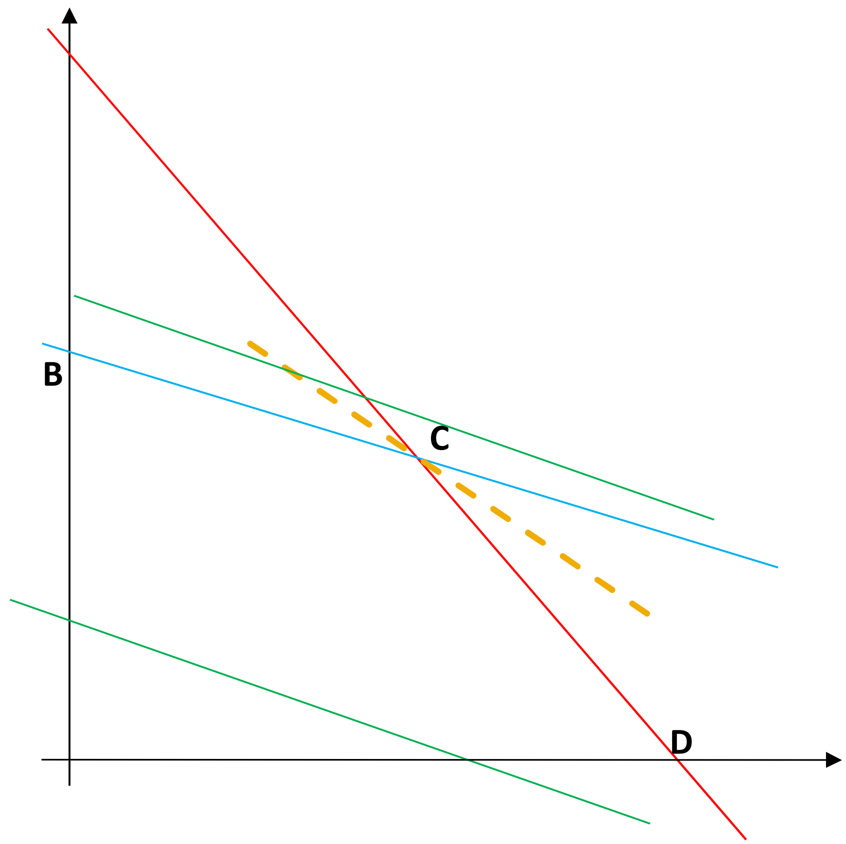


2b. New optimum with different basic variables (previous optimum is infeasible…)



3. Entire problem becomes infeasible!





A. Sensitivity to Right-Hand-Side Values

Back to our example…

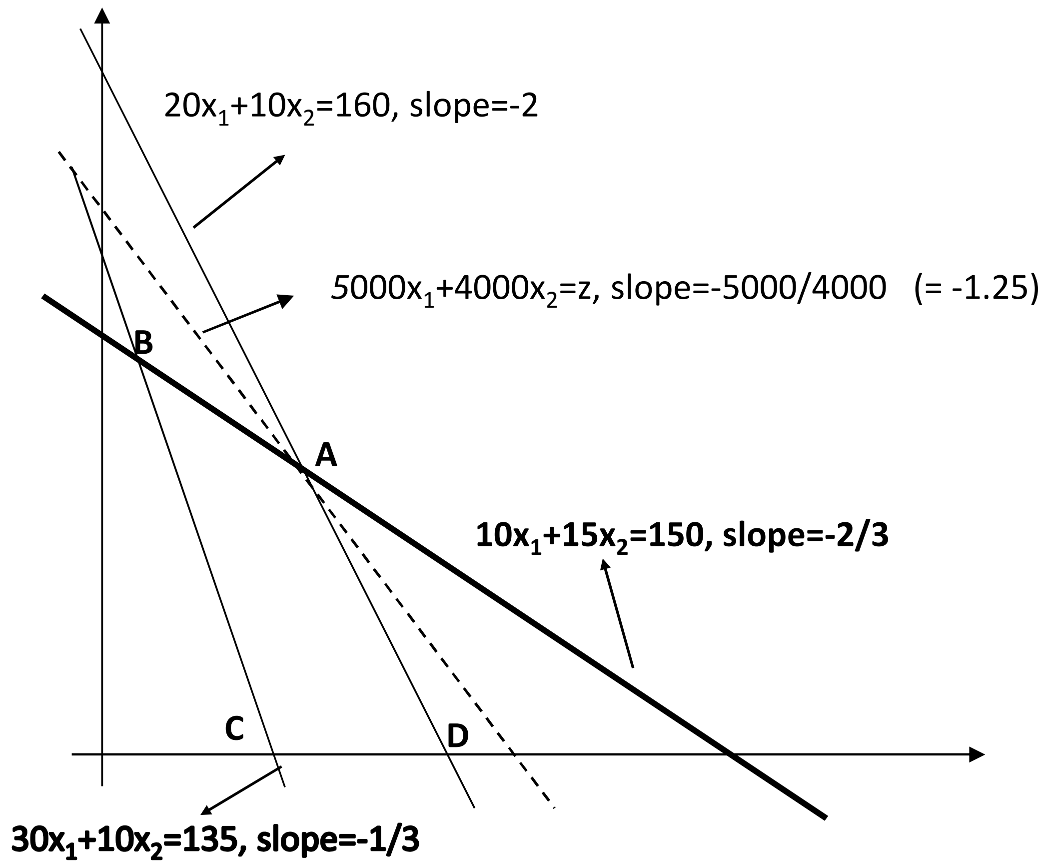
Recall that currently constraint 1 is $10x_1 + 15x_2 = 150$

We wish to examine variations in b_1 (the RHS for constraint 1) from its current value of 150; say the new value is $b_1' = b_1 + \Delta b_1$ (=150+Δb_1, in this case…).

As the value of b_1' changes the slope is unchanged but the line moves parallel to itself - either "upward" (if it increases in value) or "downward" (if it decreases in value).

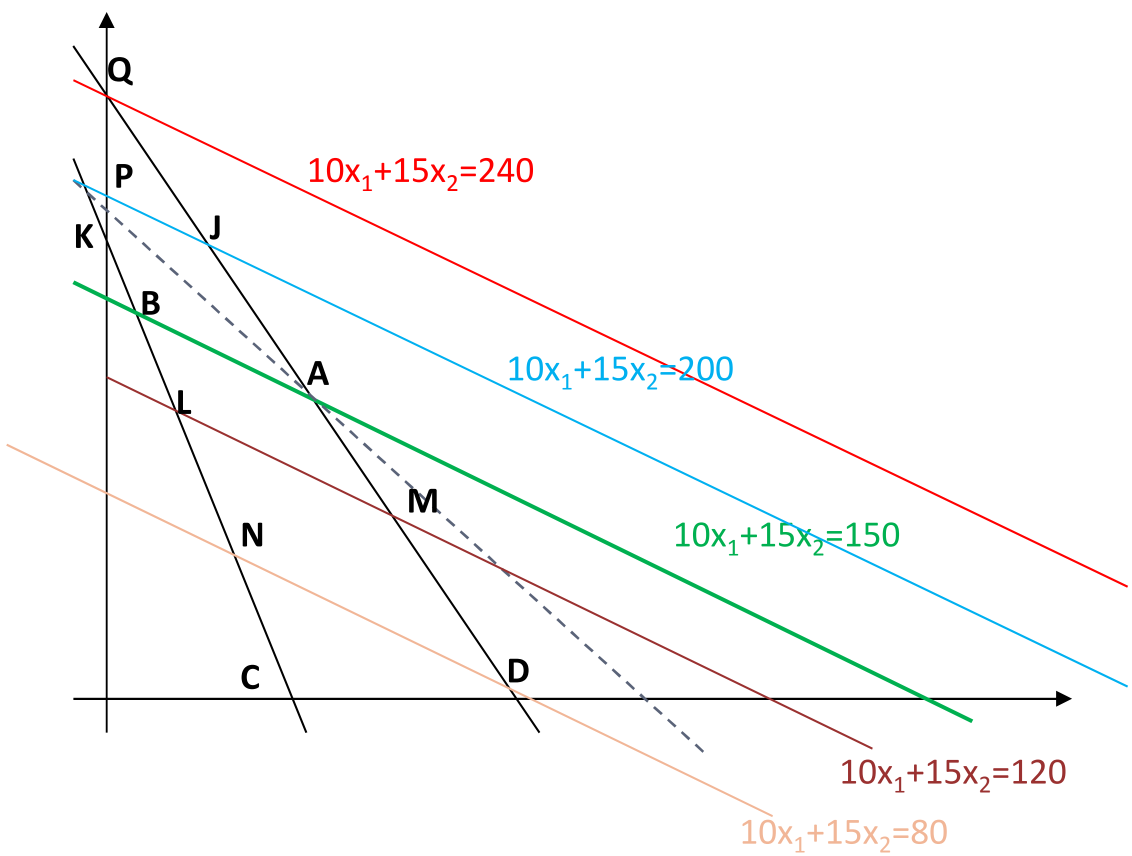
Sensitivity to the RHS values

In our example, as b_1' changes the feasible region either expands and admits more points, or shrinks and admits fewer points; in all cases the optimum solution changes as does the optimal objective value.



Sensitivity to RHS Values (cont'd)

The current feasible region is A-B-C-D



Sensitivity to RHS Values (cont'd)

Note that if

- b_1'= 120, the new feasible region is M-L-C-D and the optimum is at M (same basis)
- b_1'= 80, the new feasible region is N-C-D and the optimum is at D (different basis)

Similarly, if

- b_1'= 200, the new feasible region is J-P-K-C-D and the optimum is at J (same basis)
- b_1'= 240, the new feasible region is Q-K-C-D and the optimum is at Q (different basis)

Note that the current optimal basis is unchanged as long the new RHS satisfies

$80 \leq b_1' \leq 240$

That is, since $b_1'=150+\Delta b_1$ $\mathbf{-70 \leq \Delta b_1 \leq 90}$

Also note that the value of z^* changes in all cases (even if the basis is unchanged) and that as b_1 becomes smaller and smaller, at some point the problem could become infeasible!

Summary

- Changes in c_j do not affect feasibility in any way. However, the optimum solution may shift to another extreme point (with a different BFS) for a sufficiently large change. In either case z^* will change too.
- Changes in b_i affect the shape of the feasible region and big changes could make the feasible region vanish. If the problem is still feasible after the change the optimum may or may not change. If it does change, the value of z^* will change too and the new optimum may or may not have a different set of basic variables (i.e., may be at a point of intersection of a different set of lines than before).

LINDO and the Excel Solver provide the range of values for each c_j and for each b_i in which the basis is unchanged. They do not provide the new values of the objective though…

Ranging Output from LINDO

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	5000.000000	3000.000000	2333.333252
X2	4000.000000	3500.000000	1500.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	150.000000	90.000000	70.000000
3	160.000000	140.000000	40.000000
4	135.000000	70.000000	INFINITY

Ranging Output from Excel-Solver

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [IE1081Example.xls]Sheet4
Report Created: 10/15/2008 1:22:15 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
A1	X_1	4.5	0	5000	3000	2333.333333
A2	X_2	7	0	4000	3500	1500

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
A6	Constraint1	150	150	150	90	70
A7	Constraint2	160	175	160	140	40
A8	Constraint3	205	0	135	70	1E+30

SHADOW PRICES (*aka* DUAL PRICES)

<u>Definition</u>: The shadow price for constraint *i* is the **<u>rate</u>** at which the optimum objective z^* ***improves*** (i.e., **in**creases for a **max** problem, or **de**creases for a **min** problem) when the RHS for that constraint ($=b_i$) increases, <u>provided the basis does not change</u>.

In our example consider b_1. Suppose it becomes $b_1'=b_1+\Delta b_1$

Suppose the basis does not change so that the optimum is still at the intersection of

- the line $10x_1 + 15x_2 = 150+\Delta b_1$, and
- the line $20x_1 + 10x_2 = 160$.

This point is given by $x_1^*= 4.5-(\Delta b_1/20)$ and $x_2^*= 7+(\Delta b_1/10)$.

So the optimal objective is $z^* = 5000x_1^* + 4000x_2^*$
$= 5000\{4.5-(\Delta b_1/20)\} + 4000\{7+(\Delta b_1/10)\} = 50{,}500+\underline{150}\Delta b_1$

Thus the **shadow price for the first constraint is 150**!

SHADOW / DUAL PRICES (*cont'd*)

Similarly, the shadow price for the second constraint is equal to **175**.

LINDO (and Excel Solver) provide these dual prices as well:

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	150.000000
3)	0.000000	175.000000
4)	70.000000	0.000000

Signs of Shadow Prices

The signs of the Shadow Prices can always be predicted: suppose π_i represents the shadow price for constraint *i.* Assume that the basis is unchanged for a 1 unit increase in b_i. Recall that π_i is the improvement in the objective function for this 1 unit increase.

Case 1: Constraint i is a $\leq$ constraint.
In this case a 1 unit increase in the RHS makes it easier to satisfy, i.e., it **loosens** the constraint, i.e., **expands** the feasible region and admits more feasible points. So the new objective cannot get any worse (we have everything we had before plus additional points to choose from!). Thus the improvement is always positive, or more precisely, nonnegative:

Thus π_i must be nonnegative ($\geq$0).

Case 2: Constraint i is a $\geq$ constraint.
In this case a 1 unit increase in the RHS makes it harder to satisfy, i.e., **tightens** the constraint, i.e., **shrinks** the feasible region and eliminates some points that are currently feasible. Thus the new objective cannot get any better, (we have fewer points to choose from compared to what w e had before). Thus the "improvement" is always negative, or more precisely, nonpositive:

Thus π_i must be nonpositive ($\leq$ 0).

Case 3: Constraint i is an = constraint.
In this case π_i **could take on any sign**.

REDUCED COSTS

Recall that the reduced cost for a variable is its entry in Equation 0 in the Simplex tableau - thus the optimum reduced cost value

- for a basic variable is always equal to 0,
- for a nonbasic variable - since the tableau is optimal - is always
  ≥ 0 if we are maximizing
  ≤ 0 if we are minimizing

Also recall that the reduced cost for a nonbasic variable was defined as the decrease in z for a 1 unit increase in that variable. An alternative interpretation of the **optimum** reduced cost for a nonbasic variable x_j (currently =0 at the optimum) is as follows:

- For a **max** problem it is the required **increase** in the value of its profit coefficient c_j before it can be entered into the basis (and made positive)

- For a **min** problem it is the required **decrease** in the value of its cost coefficient c_j before it can be entered into the basis (and made positive).

LINDO and Excel Solver provide these as well:

VARIABLE	VALUE	REDUCED COST
X1	4.500000	0.000000
X2	7.000000	0.000000

COMPLEMENTARY SLACKNESS

This is an important concept that applies only to ≤ or ≥ constraints and it may be stated as follows:

At the optimum:

- if a particular inequality **constraint is non-binding** (i.e., **loose** or **inactive)** so that the corresponding slack/ excess variable is positive, then **the shadow price for that constraint must be equal to zero**, and
- If the **shadow price for some constraint is non-zero**, then that **constraint must be binding** (i.e., **tight** or **active)** so that the corresponding slack/excess variable is equal to zero.

Thus

(slack or excess variable for constraint i) * (π_i) = 0

Note that it is possible for both the slack/excess as well as the shadow price to be equal to zero - however, it is <u>impossible for both to be non-zero</u>.

DUALITY: The Primal

Consider the following LP in *n* variables and *m* constraints (we will call it a "normal" maximization problem):

<u>Program P</u> (The **Primal** LP)

Maximize $z = \sum_{j=1}^{n} c_j x_j$ st $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for $i = 1,2,\ldots,m$ $x_j \geq 0$, for $j = 1,2,\ldots,n$	i.e.,	Maximize $z = \mathbf{c}^T\mathbf{x}$ st $\mathbf{Ax} \leq \mathbf{b}$ $\mathbf{x} \geq \mathbf{0}$

Associated with this LP is another LP in *m* variables and *n* constraints (we call it a "normal" minimization problem):

DUALITY: The Dual

The Dual LP (a "normal" minimization problem) has *m* variables and *n* constraints :

Program D (The **Dual** LP)

Minimize $w = \sum_{i=1}^{m} b_i y_i$	i.e.,	Minimize $w = \boldsymbol{b}^T\boldsymbol{y}$
st		st
$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$ for $j = 1,2,\ldots, n$		$\boldsymbol{A}^T\boldsymbol{y} \geq \boldsymbol{c}$
$y_i \geq 0$, for $i = 1,2,\ldots, m$		$\boldsymbol{y} \geq \boldsymbol{0}$

The pair of programs (P and D) are referred to as a (symmetric) **Primal-Dual pair** of linear programs

The Primal-Dual Pair

Maximize $z = \sum_{j=1}^{n} c_j x_j$

st

$\sum_{j=1}^{n} a_{ij} x_j \le b_i$, for i = 1,2,…,m

$x_j \ge 0$, for j =1,2,…,n

Minimize $w = \sum_{i=1}^{m} b_i y_i$

st

$\sum_{i=1}^{m} a_{ij} y_i \ge c_j$, for j = 1,2,…,n

$y_i \ge 0$, for i =1,2,…, m

PRIMAL-DUAL PAIR: An example

Program P

Maximize $z = 50x_1+40x_2+20x_3$

st

$5x_1 + x_2 + 12x_3 \leq 25$ $\quad y_1$

$x_1 + 3x_2 + 8x_3 \leq 30$ $\quad y_2$

$x_1, x_2, x_3 \geq 0$

⇨

Program D

Minimize $w = 25y_1+30y_2$

st

$5y_1 + y_2 \geq 50$

$y_1 + 3y_2 \geq 40$

$12y_1 + 8y_2 \geq 20$

$y_1, y_2 \geq 0$

PRIMAL-DUAL PAIR: An example

<u>Program D</u>

Maximize $z = 50x_1+40x_2+20x_3$

st

$5x_1 + x_2 + 12x_3 \leq 25$

$x_1 + 3x_2 + 8x_3 \leq 30$

$x_1, x_2, x_3 \geq 0$

<u>Program P</u>

Minimize $w = 25y_1+30y_2$

st

$5y_1 + y_2 \geq 50$ $\quad x_1$

$y_1 + 3y_2 \geq 40$ $\quad x_2$

$12y_1 + 8y_2 \geq 20$ $\quad x_3$

$y_1, y_2 \geq 0$

In general, **every** linear program has another linear program associated with it – one is called the **PRIMAL** and the other is called the **DUAL**

DUALITY

1) PRIMAL has n variables $\Rightarrow$ DUAL has n constraints.

2) PRIMAL has m constraints $\Rightarrow$ DUAL has m variables.

3) Coefficient matrix for the PRIMAL is $\boldsymbol{A}$ $\Rightarrow$ Coefficient matrix for the DUAL is the <u>transpose of $\boldsymbol{A}$</u>.

4) RHS vector for the PRIMAL becomes the objective coefficient vector for the DUAL; and the objective coefficient vector for the PRIMAL becomes the RHS vector for the DUAL.

DUALITY

5) **IF** the PRIMAL is a MAXIMIZATION problem **THEN**

- First convert any '$\geq$' constraints to '$\leq$' by multiplying through by -1

a) DUAL is a MINIMIZATION problem.
b) Dual variable corresponding to a primal **'='** constraint is UNRESTRICTED.
c) Dual variable corresponding to a primal '$\leq$' constraint is NONNEGATIVE ($\geq$0)
d) Dual constraint corresponding to a nonnegative primal variable is $\geq$.
e) Dual constraint corresponding to an UNRESTRICTED primal variable is **=**.

6) **IF** the PRIMAL is a MINIMIZATION problem **THEN**

- First convert any '$\leq$' constraints to '$\geq$' by multiplying through by -1

a) DUAL is a MAXIMIZATION problem.
b) Dual variable corresponding to a primal **'='** constraint is UNRESTRICTED.
c) Dual variable corresponding to a primal '$\geq$' constraint is NONNEGATIVE ($\geq$ 0)
d) Dual constraint corresponding to a nonnegative primal variable is $\leq$.
e) Dual constraint corresponding to an UNRESTRICTED primal variable is **=**.

DUALITY: Further Notes

Always associate variables of one program with a corresponding constraint in the other.

- For a MAX problem, a $\leq$ constraint is considered "normal"
- For a MIN problem, a $\geq$ constraint is considered "normal"
- A nonnegative variable is considered "normal" for both MAX and MIN

Then

- A "normal" constraint in one problem will give rise to a (normal) nonnegative variable in the other
- Equality constraints always give rise to unrestricted variables

Similarly,

- A "normal" nonnegative variable in one problem will always give rise to a "normal" constraint in the other.
- An unrestricted variable in one will give rise to an equality constraint in the other.

DUALITY: Some Important Results

1. **The Dual of the Dual is the Primal**

2. **Symmetry:** It doesn't matter which problem is called the Primal and which one the Dual; typically we refer to the ***Primal-Dual pair***.

   Without loss of generality (from (2) above…) let us denote the Primal as the Maximization problem and the Dual as the corresponding Minimization problem, i.e.

(P)	Max	$\mathbf{c}^T\mathbf{x}$	(D)	Min	$\mathbf{b}^T\mathbf{y}$
	st	$\mathbf{A}\mathbf{x} \le \mathbf{b}$		st	$\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$
		$\mathbf{x} \ge \mathbf{0}$			$\mathbf{y} \ge \mathbf{0}$

where $\mathbf{c},\mathbf{x} \in \mathbf{R}^n$, $\mathbf{b},\mathbf{y} \in \mathbf{R}^m$, and $\mathbf{A}$ is a matrix of order $m \times n$

DUALITY: Important Results (*cont'd*)

3. **<u>Weak Duality Theorem</u>:** If the vector $\boldsymbol{x}$ is feasible in the Max problem and the vector $\boldsymbol{y}$ is feasible in the corresponding Min problem, then $\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{b}^T\boldsymbol{y}$. That is, for any two vectors that are *feasible* in their respective problems, the objective for the MAX problem is $\leq$ objective for the MIN problem.

4. **<u>Strong Duality Theorem</u>:**

   If one problem is feasible and has an optimal solution, then the other is also feasible with an optimal solution. Moreover, their optimal values are equal to each other, i.e., $\boldsymbol{c}^T\boldsymbol{x}^* = \boldsymbol{b}^T\boldsymbol{y}^*$

5. If (P) is unbounded then (D) is infeasible. If (D) is unbounded then (P) is infeasible.

6. If (P) is infeasible, then (D) is **either** unbounded **or** infeasible. If (D) is infeasible, then (P) is **either** unbounded **or** infeasible.