

Axiomatic Method

- A procedure to prove results (theorems). Results often initially obtained by experimentation, observation, trial and error or “intuitive insight.”
- **Note:** We use standard **2-value logic**, that is a statement is either true or false to prove our results.

Logical Cycle

- A logical system is based upon a hierarchy of **statements**.
- Our statements consist of **terms**.
- The terms are based upon **definitions**.
- Definitions utilize new terms.
- The new terms are given definitions.
- These definitions use more new terms (or they are based upon previous terms).
- Thus, we either create an infinite chain of term-def-term-def- or we create a logical cycle.

Starting Place

- In order to provide a sound base for our logical system, we must provide a starting place.
- **Undefined terms:** used to avoid a logical cycle and the infinite digression.
- **Axioms:** initial statements which are accepted without justification.

Fe-Fo Example

- Undefined terms: Fe’s, Fo’s, and the relation “belongs to.”
- Axiom 1: There exists exactly 3 distinct Fe’s in the system.
- Axiom 2: Any two distinct Fe’s belong to exactly one Fo.
- Axiom 3: Not all Fe’s belong to the same Fo.
- Axiom 4: Any two distinct Fo’s contain at least one Fe that belongs to both.

Fe-Fo Results

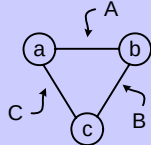
- Theorem 1: Two distinct Fo’s contain exactly one Fe.
- Theorem 2: There are exactly 3 Fo’s.
- Theorem 3: Each Fo has exactly two Fe’s that belong to it.

Axiomatic Applications

- **Interpretation:** provide a “real” meaning to the axiomatic system.
- **Model:** an interpretation that satisfies all the axioms of the system.
- Fe-Fo Model 1 (Graph)
Fe: node (vertex) Fo: edge
Belongs: adjacent to

Fe-Fo Model 1 (Graph)

- Axiom 1: There exists exactly 3 distinct nodes.
- Axiom 2: Any two distinct nodes are contained in exactly one edge.
- Axiom 3: Not all nodes belong to the same edge.
- Axiom 4: Any two distinct edges contain at least one node that belongs to both.

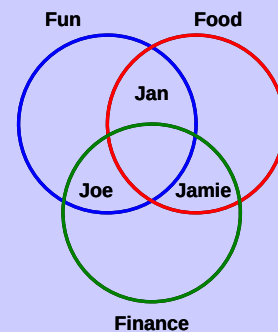


Axiomatic Applications

- **Interpretation:** provide a “real” meaning to the axiomatic system.
- **Model:** an interpretation that satisfies all the axioms of the system.
- Fe-Fo Model 1 (Graph)
Fe: nodes (vertices) Fo: edges
Belongs: adjacent to
- Fe-Fo Model 2 (Committee)
Fe: person Fo: committees
Belongs: a member of

Fe-Fo Model 2 (Committee)

- Axiom 1: There exists exactly 3 distinct people.
- Axiom 2: Any two distinct people are members of exactly one committee.
- Axiom 3: Not all people are members of the same committee.
- Axiom 4: Any two distinct committees contain at least one person that is a member of both committees.



Axiomatic Applications

- Fe-Fo Model 1 (Graph)
Fe: node (vertex) Fo: edge
Belongs: adjacent to
- Fe-Fo Model 2 (Committee)
Fe: person Fo: committee
Belongs: a member of
- Fe-Fo Model 3 (Bookshelf)
Fe: book Fo: shelf
Belongs: is on

Fe-Fo “Model” 3 (Bookshelf)

- Axiom 1: There exists exactly 3 distinct books.
- Axiom 2: Any two distinct books are members of exactly one shelf.
- Axiom 3: Not all books are on of the same shelf.
- Axiom 4: Any two distinct shelves there is at least one book that is on both shelves.
- This interpretation is **NOT** a model.

Consistent Axiom Sets

- An axiom set is said to be **consistent** if it is impossible to deduce from it a theorem that contradicts an axiom or another deduced theorem.
- An axiom set is said to have **absolute consistency** if there exists a real world model satisfying all of the axioms.
- An axiom set is said to be **relatively consistent** if we can produce a model for the axiom set based upon another axiom set which we are willing to assume is consistent.

Consistent Axiom Sets

- Example:
Undefined terms: Hi, Lo and belongs to.
Axiom 1: There are exactly 4 Hi's.
Axiom 2: Every Hi belongs to exactly two Lo's.
Axiom 3: Any two Hi's belong to at most one Lo.
Axiom 4: There is a Lo containing any two Hi's.
Axiom 5: All Lo's contain exactly two Hi's.
- This is an **inconsistent** system.

Absolute Consistent Axiom Set

- Example:
The Fe-Fo Axiom Set exhibits absolute consistency because we produced a real world model for the system (i.e. actually two, the committee model and the graph model).
- Note: It is true that we also produced a "non-model" (the books-shelves model) but this does not imply the system is not consistent.

Relative Consistency

- Example: (Real Numbers)
We can not produce a concrete, real-world model (we only have a finite number of objects to manipulate). If we then show that the real numbers are a model for Axiom Set A then we say Axiom Set A is relatively consistent

Real Number Axioms

- I. Field Axioms (additive axioms, multiplicative axioms, distributive laws)
- II. Order Axioms (trichotomy, transitivity, additive compatibility, multiplicative compatibility)
- III. Least Upper Bound Axioms

Real Number Axioms - Field Axioms

- **Additive Axioms:**
 $x + y \in \mathbf{R}$ $x + y = y + x$
 $(x + y) + z = x + (y + z)$ $x + 0 = 0 + x$
 $x + (-x) = (-x) + x = 0$
- **Multiplicative Axioms:**
 $xy \in \mathbf{R}$ $xy = yx$
 $(xy)z = x(yz)$ $x1 = 1x = x$
 $x(x^{-1}) = (x^{-1})x = 1$ (if $x \neq 0$)
- **Distributive Axioms:**
 $x(y + z) = xy + xz$ $(y + z)x = (yx + zx)$

Real Number Axioms - Order Axioms

- **Trichotomy:**
Either $x = y$, $x > y$ or $x < y \forall x, y \in \mathbf{R}$.
- **Transitivity:**
For $x, y, z \in \mathbf{R}$, if $x > y$ and $y > z$ then $x > z$.
- **Additive Compatibility:**
For $x, y, z \in \mathbf{R}$, if $x > y$ then $x + z > y + z$.
- **Multiplicative Compatibility:**
For $x, y, z \in \mathbf{R}$, if $x > y$ and $z > 0$ then $xz > yz$.

Real Number Axioms - Least Upper Bound

- **Definitions:**
A number M is said to be an **upper bound** for a set X , $X \subseteq \mathbf{R}$, if $x < M \forall x \in X$.
A number M is said to be a **least upper bound** for a set X , denoted $\text{lub}(X)$, if it is an upper bound of X and $M < N$ for all other upper bounds of X .
- **Least Upper Bound Axiom:** If a set X has an upper bound, then it has a least upper bound.
- **Note:** This is also called the **Dedekind Completeness Axiom**.

Axiom Independence

- **Definitions:**
An axiom is said to be **independent** if that axiom can not be deduced as a theorem based solely on the other axioms.
If all axioms are independent then the axiom set is **independent**.
- **Note:** If you can produce a model whereby all the axioms hold except one, then that lone axiom is independent of the others.

Fe-Fo Example

- **Independence of Axiom 1**
Axiom 1: There exists exactly 3 distinct Fe's in the system.
Axiom 2: Any two distinct Fe's belong to exactly one Fo.
Axiom 3: Not all Fe's belong to the same Fo.
Axiom 4: Any two distinct Fo's contain at least one Fe that belongs to both.

Fe: a,b,c,d
Fo: line segments

