[redacted]

1. Let X be an n-dimensional vector space. Prove that $\mathrm{Hom}(X, X)$ has dimension n^2.

2. Let $T : P_n \to P_n$ be the linear map $Tp(t) = p(t+1)$ and let D be the differentiation operator. Show that

$$T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^{n-1}}{(n-1)!}$$

3. Let X be an n-dimensional vector space and $A : X \to X$ a linear operator. Show that there is a non-zero polynomial $p(t)$ of degree $\leq n^2$ such that $p(A) = 0$.

4. Let V be an n-dimensional vector space and suppose that $T \in \mathrm{End}(V)$ satisfies $T^n = 0$ and $T^{n-1} \neq 0$. Prove that there exists a vector $x \in V$ such that $\{x, Tx, T^2x, \ldots, T^{n-1}x\}$ is a basis of V. What is the matrix of T in this basis?

5. Show that there exist integers a, b, c, d such that the following two matrices are similar over $\mathbb{Q}$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \end{bmatrix}.$$

Find these integers.

6. Prove: If A is a linear operator such that $A^2 - A + I = 0$, then A is invertible.

7. Prove that there are no square matrices A, B such that $AB - BA = I$.

8. If A, B, C are linear operators on a two-dimensional vector space, show that $(AB - BA)^2$ commutes with C.

9. A projection on a vector space V is an operator such that $A^2 = A$. If A is a projection, show that either $A = 0$, $A = I$, or there is a basis of V such that the representation of A with respect to B has block form $[S]_B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

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1. Find all of the eigenvalues and eigenvectors of the backward shift operator on $\mathbb{C}^\infty$ given by

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots).$$

2. Show that an $n \times n$ matrix A is never similar to $A + I$.

3. A matrix A is called skew-symmetric if $A^T = -A$. Let A be an $n \times n$ skew-symmetric matrix with n odd.

(a) Show that $\det A = 0$.

(b) Show that all of the nonzero eigenvalues of A are imaginary.

4. Find two non-similar 4×4 matrices with characteristic polynomial $(x-2)^4$ and minimal polynomial $(x-2)^2$.

5. If the $n \times n$ matrix $I - AB$ is invertible, show that $I - BA$ is inveritble.

6. Suppose that $S, T \in \operatorname{Hom}(V, V)$ with V finite dimensional.

(a) Suppose that $\dim \operatorname{im} T = k$. Show that T has at most $k+1$ distinct eigenvalues.

(b) Show that ST and TS have the same eigenvalues.

(c) Show that if every vector in V is an eigenvector of S, then $S = aI$.

7. Suppose that $T \in \operatorname{Hom}(V, V)$ is such that every subspace of dimension $\dim V - 1$ is invariant under V. Prove that T is a scalar multiple of the identity.

8. Find the minimal polynomial of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

9. Let V be a finite dimensional complex vector space and $T \in \operatorname{Hom}(V, V)$ be a map such that the kernel and image satisfy

$$\ker(T - \lambda I) \cap \operatorname{im}(T - \lambda I) = \{0\} \qquad \text{for all } \lambda \in \mathbb{C}.$$

Show that there is a basis of V consisting of eigenvectors of T.

10. Let A be a 3×2 matrix and B be a 2×3 matrix such that

$$AB = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}.$$

Find *with proof* BA.

[redacted]

1. Let P_5 be the space of complex polynomials of degree less than 5. Let D be the differentiation operator on P_5. Find the eigenvalues, eigenvectors, characteristic polynomial, and minimal polynomial of D.

2. What is the minimal polynomial of:
(a) a projection?
(b) an involution?
(c) the map T on P_n such that $T(p(t)) = p(t+1)$?

3. True or false: If A is a real 2×2 matrix with eigenvalues $a \pm ib$, then A is similar to $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. (If true, prove; if false, provide a counterexample.)

4. Under what conditions on the complex numbers $a_1, a_2, \ldots, a_n$ is the following matrix diagonalizable over $\mathbb{C}$?

$$\begin{bmatrix} 0 & \cdots & 0 & a_1 \\ 0 & \cdots & a_2 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ a_n & \cdots & 0 & 0 \end{bmatrix}$$

5. Suppose that $T \in L(V,V)$ has a cyclic vector (i.e., there is a vector $x \in V$ such that $x, Tx, T^2x, \ldots, T^{n-1}x$ is a basis of V). Show that if $U \in L(V,V)$ and $UT = TU$ then U is a polynomial in T.

6. Suppose that $T \in L(V,V)$ and $\dim R_T = 1$. Show that T is either diagonalizable or nilpotent (but not both).

7. Let A and B be linear operators on a finite-dimensional complex vector space V. Let p be any polynomial such that $p(AB) = 0$.
(a) Show that if $q(s) = sp(s)$, then $q(BA) = 0$.
(b) Use the result of (a) to show that the minimal polynomials m_{AB} and m_{BA} obey either $m_{AB}(s) = m_{BA}(s)$, $m_{AB}(s) = sm_{BA}(s)$, or $m_{BA}(s) = sm_{AB}(s)$.

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In the following exercises, you will find the Jordan decompositions of various matrices. That is, you will find a matrix S and a Jordan matrix J such that $A = SJS^{-1}$.

These *can* be done by hand! Do not use a computer to do the problems, but you may use one to check yourself.

1. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the Jordan decomposition of A.

2. Find the Jordan decomposition of

$$A = \begin{bmatrix} 0 & 0 & 4 & 4 \\ -1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 4 \end{bmatrix}.$$

3. The characteristic polynomial of

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & -1 \\ -1 & -2 & 2 & 4 & 3 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

is $p(x) = -(x+2)^5$. Find the Jordan decomposition of A.