MATH3650: Symplectic Geometry

Take-Home Final Exam

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Due: by Tuesday morning (December 10, 2013).

You only need to do 4 out of 5 problems, although you will receive extra points for doing all of them.

Problem 1: Let H denote the subgroup of $\mathrm{GL}(3, \mathbb{R})$ consisting of the matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) What is the Lie algebra of H (as a subalgebra of $\mathfrak{gl}(3, \mathbb{R}) = \mathrm{Mat}(3, \mathbb{R})$)? (You don't need to check that H is a closed subgroup of $\mathrm{GL}(3, \mathbb{R})$.)

(b) Let $\exp : \mathrm{Lie}(H) \to H$ be the exponential map of H. For $\xi \in \mathrm{Lie}(H)$ find $\exp(\xi)$ in terms of entries of ξ. (Recall that the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n)$ is given by exponential of matrices $\exp(A) = \sum_{k=0}^{\infty} A^k/k!$.

(c) Consider the linear action of H on $\mathbb{R}^3$. Let $\xi \in \mathrm{Lie}(H)$ be a Lie algebra element. Write down a formula for the generating vector field of ξ (as a vector field on $\mathbb{R}^3$).

Problem 2: Let (M, ω) be a symplectic manifold with a Hamiltonian action of a compact Lie group G with moment map $\mu_G : M \to \mathrm{Lie}(G)^*$. Let $H \subset G$ be a closed subgroup of G.

(a) Show that action of H on M is also Hamiltonian and its moment map $\mu_H : M \to \mathrm{Lie}(H)^*$ is given by $\mu_H = \pi \circ \mu_G$ where $\pi : \mathrm{Lie}(G)^* \to \mathrm{Lie}(H)^*$ is the linear projection induced by the inclusion $\mathrm{Lie}(H) \hookrightarrow \mathrm{Lie}(G)$.

(b) Suppose moreover that H is a normal subgroup. Then one knows that G/H is a Lie group. Also assume that $0 \in \mathrm{Lie}(H)^*$ is a regular value for μ_H, and H acts freely on $\mu_H^{-1}(0)$. Show that the H-reduced space

$M_{red} := \mu_H^{-1}(0)/H$ (as usual equipped with Marsden-Weinstein-Meyer symplectic form) is a Hamiltonian space for the action of G/H and its moment map is given by $j \circ \mu_G$ where $j : \mathrm{Lie}(G/H)^* \hookrightarrow \mathrm{Lie}(G)^*$ is the embedding induced by the natural projection $\mathrm{Lie}(G) \to \mathrm{Lie}(G/H)$.

Problem 3: Consider the linear action of the 2-dimensional complex torus $T_{\mathbb{C}} = (\mathbb{C}^*)^2$ on $\mathbb{C}^6$ and its induced action on $\mathbb{CP}^5$, by

$$(t_1, t_2) \cdot (z_1 : z_2 : z_3 : z_4 : z_5 : z_6) = (t_1^2 z_1 : t_1 t_2 z_2 : t_2^2 z_3 : t_1 z_4 : t_2 z_5 : z_6).$$

(a) Write down formulae for the moment maps $\psi : \mathbb{C}^6 \to \mathbb{R}^2$ and $\mu : \mathbb{CP}^5 \to \mathbb{R}^2$ for the corresponding $T = (S^1)^2$ actions.

(b) Find the images of the moment maps ψ and μ.

(c) Let Y be the $T_{\mathbb{C}}$-orbit of the point $(1 : 1 : 1 : 1 : 1 : 1) \in \mathbb{CP}^5$ an let $\overline{Y}$ be the closure of Y. Find the image Δ of moment map μ restricted to $\overline{Y}$.

(d) Find the inverse image (under $\mu_{|\overline{Y}}$) of the vertices of Δ. Show that they are exactly the fixed points for action of $T_{\mathbb{C}}$ on M.

Problem 4: Let $\Omega(V)$ and $J(V)$ be the spaces of symplectic forms and complex structures on the real vector space $V = \mathbb{R}^{2n}$ respectively. Let $\Omega \in \Omega(V)$ and $J \in J(V)$ be the standard symplectic form and standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$. Let $\mathrm{GL}(2n, \mathbb{R})$ be the group of all linear isomorphisms of V, $\mathrm{Sp}(2n)$ the group of symplectomorphisms of $(\mathbb{R}^{2n}, \Omega)$, and $\mathrm{GL}(n, \mathbb{C})$ the group of complex linear isomorphisms of $V = \mathbb{C}^n$. Show that $\Omega(V) \cong \mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n)$ and $J(V) \cong \mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$. (Hint: The group $\mathrm{GL}(2n, \mathbb{R})$ acts on $\Omega(V)$ by pullback. Show that the action is transitive. What is the stabilizer of Ω?)

Problem 5: Consider the compact Lie group $G = \mathrm{SU}(3)$. Recall that $\mathfrak{su}(3) = \mathrm{Lie}(\mathrm{SU}(3))$ is the collection of all skew-Hermitian 3×3 matrices with trace 0 (a matrix A is skew-Hermitian $\Leftrightarrow \overline{A}^t = -A \Leftrightarrow iA$ is Hermitian). Also $\mathfrak{su}(3)^*$ can be identified with $\mathfrak{su}(3)$ via the inner product (Killing form) $\langle A, B \rangle = \mathrm{tr}(AB)$. Take $\xi \in \mathfrak{su}(3)^*$. Let $\mathcal{O} = G \cdot \xi$ be the coadjoint orbit of ξ. Show that $\mathcal{O}$ can be identified with set of Hermitian matrices with given real eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Consider $\mathcal{O}$ as a symplctic manifold with the Kostant-Kirillov symplectic form. Let T be the subtorus of G consisting of diagonal matrices. Show that the action of T on $\mathcal{O}$ is Hamiltonian and find its moment map (Hint: Problem 2). (The well-known Horn-Shur theorem in representation theory describes the image of this moment map.)