MATH 2370 (Fall 2015)
Final exam information sheet

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Summary of topics:

- Vector space, subspace, independent set, spanning set, basis, isomorphism of vector spaces, a basis for an n-dimensional vector space V gives an isomorphism between V and K^n (where K is the base field).

- Direct sum of two vector spaces, direct sum of two subspaces of a vector space.

- Linear map, matrix of a linear map with respect to a basis, change of basis matrix, the matrices M_1, M_2 of a linear map T with respect to two different bases B_1, B_2 are similar matrices.

- If $V = W \oplus U$ then we have a projection $P : V \to W$ (projection parallel to U), conversely if P is such that $P^2 = P$ (i.e. a projection) then $V = R_P \oplus N_P$.

- Dual V' of a vector space V, dual T' of a linear map T. Dual of a linear map corresponds to the transpose of the corresponding matrix.

- There is a natural pairing $V \times V' \to K$ between V and it dual space V'. It is a bilinear map. If B is a basis for V there is a natural dual basis B' for V' called the dual basis such that the pairing between the elements of B and B' is the identity matrix.

- Nilpotent matrix, permutation matrix.

- Determinant, geometric interpretation of determinant as (signed) volume, definition of determinant as an alternating n-linear map, formula for determinant in terms of sum over all permutations, determinant of special matrices: diagonal, upper/lower triangular, block diagonal, Cramer's rule for solving a system of equations using determinants, formula for inverse of a matrix in terms of determinants, Laplace row/column expansion of a determinant, $det(A) = det(A^T)$, trace, $tr(AB) = tr(BA)$, trace is invariant under conjugation.

- Eigenvalues, eigenvectors, generalized eigenvectors, characteristic polynomial, minimal polynomial, first and last coefficients of characteristic polynomial in terms of trace and determinant.

- Spectral Theorem, how to find (generalized) eigenvectors and eigenvalues of a give matrix, index of a generalized eigenvector, Jordan decomposition
- How to find Jordan form if the characteristic polynomial and the minimal polynomial are given.
- A and A^T are similar matrices.
- Bilinear form $B : V \times V \to K$ (where K is the base field), matrix M of a bilinear form B with respect to a choice of a basis for V, if we change the basis then the matrix M of B changes to $S^T MS$ where S is the change of basis matrix.
- Euclidean structures, scalar product, orthonormal basis, Gram-Schmidt, orthogonal matrix
- Adjoint linear map, orthogonal projection, reflection with respect to a hyperplane (formula).
- Theorem of least squares, i.e. given matrix A and vector p find x such that $||Ax - p||^2$ is minimum.
- Hermitian structures, Hermitian product, Hermitian matrix, unitary matrix
- Operator norm, Hilbert-Schmidt norm, spectral radius, Gelfand's formula
- Theorem that if A is invertible and $||A - B|| < 1/||A^{-1}||$ then B is invertible.
- Self-adjoint operator, normal matrix, every normal matrix is orthogonally diagonalizable
- Quadratic form, matrix of a quadratic form, diagonalizing a quadratic form by a linear change of variables, how to distinguish that a curve defined in the plane by a quadratic equation is an ellipse or hyperbola
- Sylvester's law of inertia about number of positive, negative and zero eigenvalues of a symmetric real matrix (or its associated quadratic form)
- Courant's Minimax Principle (Theorem) about the i-th eigenvector of a Hermitian matrix.

About the final:

- The final is 1 hour and 50 minutes long.
- There will be $5 - 6$ questions (each possibly broken into few parts).
- Please make sure you can do all the homework problems.

- One question asks to state some definitions and state a theorem (discussed in class) without proof.

- There will be a proof question from theorems in class (easier ones). Some examples are as follows: (1) Show that the eigenvectors of a Hermitian matrix (corresponding to different eigenvalues) are orthogonal. (2) Show that if $AB = BA$ and A, B are diagonalizable then A and B can be simultaneously diagonalized. (3) Show that the eigenvalues of a Hermitian matrix are real.