A Larger Example of the Galois Correspondence

Let E be the splitting field of the polynomial $f = x^4 - 2$ over $\mathbb{Q}$.

1. Show that $E = \mathbb{Q}(\alpha, i)$ where $\alpha = \sqrt[4]{2}$.

2. Prove that $(E : \mathbb{Q}) = 8$.

3. Show that the Galois group $G = \Gamma(E : \mathbb{Q})$ is generated by the automorphisms σ and τ where
$$\sigma : \ \alpha \mapsto i\alpha, \ \ i \mapsto i,$$
$$\tau : \ \alpha \mapsto \alpha, \ \ i \mapsto -i.$$
(Hint: Show that σ has order 4, τ has order 2, that $\tau \notin \langle\sigma\rangle$, and deduce that each element of G is of the form $\sigma^i\tau^j$ with $i \in \{0, 1, 2, 3\}$ and $\tau \in \{0, 1\}$).

4. Make a table showing the effect of the 8 automorphisms $\sigma^i\tau^j$ with $i \in \{0, 1, 2, 3\}$ and $\tau \in \{0, 1\}$ in G on the elements α and i.

5. Prove that
$$(\sigma^i\tau^j)(\sigma^k\tau^l) = \sigma^{i+(-1)^j k}\tau^{j+l}$$
where the exponents at σ and τ are understood to be taken modulo 4 and 2, respectively.

6. Determine the orders of all elements of G.

7. Make a list of all cyclic subgroups of G.

8. Show that the centre of G is the cyclic subgroup $Z(G) = \langle\sigma^2\rangle$.

9. Show that the only proper non-cyclic subgroups of G are
$$\langle\sigma^2\rangle \times \langle\tau\rangle = \{1, \sigma^2, \tau, \sigma^2\tau\}$$
and
$$\langle\sigma^2\rangle \times \langle\tau\sigma\rangle = \{1, \sigma^2, \sigma\tau, \sigma^3\tau\}.$$

10. Show that
    (a) $\Phi(\langle\sigma\rangle) = \mathbb{Q}(i)$,
    (b) $\Phi(\langle\sigma^2\rangle \times \langle\tau\rangle) = \mathbb{Q}(\sqrt{2})$,
    (c) $\Phi(\langle\sigma^2\rangle \times \langle\tau\sigma\rangle) = \mathbb{Q}(i\sqrt{2})$.

11. Show that $Z(G) = \langle\sigma^2\rangle$ is the only normal subgroup of order 2 in G.

12. Show that

(a) $\Phi(\langle \tau \rangle) = \mathbb{Q}(\alpha)$,

(b) $\Phi(\langle \sigma^2 \rangle) = \mathbb{Q}(i, \sqrt{2})$,

(c) $\Phi(\langle \sigma\tau \rangle) = \mathbb{Q}((1+i)\alpha)$.