1. Let E be the splitting field inside $\mathbb{C}$ of the polynomial $f = x^5 - 2$, and let $G = \Gamma(E : \mathbb{Q})$.

   (a) Show that $E = \mathbb{Q}(\alpha, \epsilon)$ where $\alpha = \sqrt[5]{2}$ and $\epsilon = \exp(\frac{2\pi i}{5})$.

   (b) Show that $(E : \mathbb{Q}) = 20$, and deduce that $|G| = 20$.

   (c) Show that G is generated by the automorphisms θ and σ given by

   $$\theta(\alpha) = \epsilon\alpha, \qquad \theta(\epsilon) = \epsilon,$$

   $$\sigma(\alpha) = \alpha, \quad \sigma(\epsilon) = \epsilon^2.$$

   (Hint: Determine all possible images of α and ϵ under an automorphisms in G, deduce that each of those possibilities gives rise to an actual automorphism, and show that each such automorphism is of the form $\theta^i\sigma^j$ with $0 \leq i \leq 4$ and $0 \leq j \leq 3$.

   (d) Verify that θ generates a cyclic subgroup of order 5, and that σ generates a cyclic subgroup of order 4 in G.

   (e) Work out the conjugate $\sigma\theta\sigma^{-1}$ and deduce that the cyclic subgroup $\langle\theta\rangle$ is normal in G.

   (f) Find the fixed fields for the normal subgroup $\langle\theta\rangle$, for the cyclic subgroup $\langle\sigma\rangle$ and for the subgroup generated by θ and σ^2.

2. Let E be the splitting field inside $\mathbb{C}$ of the polynomial $f = x^6 - 2$, and let $G = \Gamma(E : \mathbb{Q})$.

   (a) Show that $E = \mathbb{Q}(\alpha, \epsilon)$ where $\alpha = \sqrt[6]{2}$ and $\epsilon = \exp(\frac{2\pi i}{6})$.

   (b) Show that $(E : \mathbb{Q}) = 12$, and deduce that $|G| = 12$.

   (c) Show that G is generated by the automorphisms θ and σ given by

   $$\theta(\alpha) = \epsilon\alpha, \qquad \theta(\epsilon) = \epsilon,$$

   $$\sigma(\alpha) = \alpha, \quad \sigma(\epsilon) = \bar{\epsilon} = \epsilon^5.$$

   (Hint: Determine all possible images of α and ϵ under an automorphisms in G, deduce that each of those possibilities gives rise to an actual automorphism, and show that each such automorphism is of the form $\theta^i\sigma^j$ with $1 \leq i \leq 6$ and $0 \leq j \leq 1$.

   (d) Verify that θ generates a cyclic subgroup of order 6, and that σ generates a cyclic subgroup of order 2 in G.

   (e) Work out the conjugate $\sigma\theta\sigma^{-1}$. Deduce that G is isomorphic to the dihedral group of order 12.

(f) Find the fixed fields for the normal subgroup $\langle\theta\rangle$, for the cyclic subgroup $\langle\sigma\rangle$ and for the cyclic subgroup $\langle\theta^3\rangle$.

3. Let E be the splitting field inside $\mathbb{C}$ of the polynomial $f = x^8 - 2$, and let $G = \Gamma(E : \mathbb{Q})$.

   (a) Show that $E = \mathbb{Q}(\alpha, \epsilon)$ where $\alpha = \sqrt[8]{2}$ and $\epsilon = \exp(\frac{2\pi i}{8})$.

   (b) The degree $(E : \mathbb{Q})$ is 32.

   Proof: The field E can be obtained from $\mathbb{Q}$ by two consecutive extensions: by adjoining first $\sqrt[8]{2}$ and then ϵ. The miminum polynomial of $\sqrt[8]{2}$ over $\mathbb{Q}$ is $x^8 - 2$. Hence $(\mathbb{Q}(\alpha) : \mathbb{Q}) = 8$. The minimum polynomial of ϵ over $\mathbb{Q}$ is $x^4 + 1$, and this stays irreducible over $\mathbb{Q}(\alpha)$ since all its roots are non-real complex numbers. Hence $(E : \mathbb{Q}(\alpha)) = 4$, and then the tower law gives that $(E : \mathbb{Q}) = 32$.

   This, however, is not true. So what is wrong with the proof?

   (c) Show that $E = \mathbb{Q}(\alpha, i)$ where $\alpha = \sqrt[8]{2}$.

   (d) Show that $(E : \mathbb{Q}) = 16$, and deduce that $|G| = 16$.

   (e) Show that G contains an automorphism θ such that

   $$\theta(\alpha) = \epsilon\alpha, \qquad \theta(i) = i,$$

   where $\epsilon = \exp(\frac{2\pi i}{8})$, and an automorphism σ such that

   $$\sigma(\alpha) = \alpha, \quad \sigma(i) = -i.$$

   (f) Show that $\theta(\epsilon) = -\epsilon$.

   (g) Work out $\theta^i(\alpha)$ for $i = 1, 2, \ldots, 8$ and hence verify that θ generates a cyclic subgroup of order 8.

   (h) Verify that σ generates a cyclic group of order 2.

   (i) Work out the conjugate $\sigma\theta\sigma^{-1}$, and determine the conjugation action of σ on the cyclic subgroup $\langle\theta\rangle$.

   (j) Show that G is *not* isomorphic to the dihedral group D_{16} of order 16. (Hint: Work out the order of $\theta\sigma$.)

   (k) Find the fixed fields for the cyclic subgroups $\langle\theta\rangle$, $\langle\theta^2\rangle$, $\langle\theta^4\rangle$ and $\langle\sigma\rangle$, and for the subgroup H that is generated by θ^4 and σ.

4. (Something of a curiosity) Prove that the Galois group $\Gamma(\mathbb{R} : \mathbb{Q})$ is trivial. (Hint: Show that any $\mathbb{Q}$-automorphisms of $\mathbb{R}$ maps positive real numbers to positive real numbers (for $\alpha > 0$, consider $\sqrt{\alpha}$), deduce that any $\mathbb{Q}$-automorphism preserves the natural order relation "$\geq$" on $\mathbb{R}$, then take any real number α and two sequences of rational numbers, one monotonic increasing, the other monotonic decreasing and both converging to α, and argue that α must be fixed.)