1. The roots of f are $\pm\sqrt[4]{2}$ and $\pm i\sqrt[4]{2}$. Hence
$$\mathbb{Q}(\pm\sqrt[4]{2}, \pm i\sqrt[4]{2}) = \mathbb{Q}(\sqrt[4]{2}, i) = \mathbb{Q}(\alpha, i)$$
is a splitting field of f inside $\mathbb{C}$.

2. The splitting field E can be obtained from $\mathbb{Q}$ by two consecutive simple extension, namely by first adjoining α and then adjoining i. The minimum polynomial of α over $\mathbb{Q}$ is $x^4 - 2$. Indeed, α is a zero and the polynomial is irreducible by Eisenstein's criterion. Hence $(\mathbb{Q}(\alpha) : \mathbb{Q}) = 4$. The minimum polynomial of i over $\mathbb{Q}(\alpha)$ is $x^2 + 1$. Indeed, i is a zero of this polynomial, and it is irreducible over $\mathbb{Q}(\alpha)$ as it has no zero in $\mathbb{Q}(\alpha)$ which consists entirely of real numbers. Hence $(\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)) = 2$. Now the Tower Law, applied to $\mathbb{Q} \subseteq \mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\alpha, i) = E$ gives $(E : \mathbb{Q}) = 8$.

3. Since E is a splitting field in characteristic zero, $E : F$ is a normal extension, and the Fundamental Theorem of Galois Theory applies to it. In particular, the order of the Galois group G is 8. Since E is generated by α and i, each $\mathbb{Q}$-automorphism of E is completely determined by its effect on α and i. Since any such automorphism maps α to a zero of $x^4 - 2$, there are 4 possible images for α, namely $\pm\alpha$, $\pm i\alpha$. Also, since any $\mathbb{Q}$-automorphism maps i to a zero of $x^2 + 1$, the possible images for i are $\pm i$. Hence there are precisely 8 possible images for the pair α, i and since the order of G is 8 all 8 of these possibilities give us actual automorphisms in G. Of course, $\alpha \mapsto \alpha$ and $i \mapsto i$ gives the 1 of G. Let σ denote the automorphism given by $\alpha \mapsto i\alpha$ and $i \mapsto i$. Then i is fixed by all powers of σ, and for α we have
$$\sigma(\alpha) = i\alpha, \qquad \sigma^2(\alpha) = -\alpha, \qquad \sigma^3(\alpha) = -i\alpha, \qquad \sigma^4(\alpha) = \alpha.$$
Hence $\sigma^4 = 1$ and σ generates a cyclic group of order 4 in G. Let τ denote the automorphisms given by $\alpha \mapsto \alpha$ and $i \mapsto -i$. Clearly, $\tau \notin \langle\sigma\rangle$, and $\tau^2 = 1$. Consequently, the products $\sigma^i\tau$ with $(i = 0, 1, 2, 3)$ together with the elements of $\langle\sigma\rangle = \{\sigma^i;\ i = 0, 1, 2, 3\}$ give 8 distinct elements in G, that is all elements of G.

4. The 8 elements of G and their effect on α and i is given in the following table

auto	α	i
1	α	i
σ	$i\alpha$	i
σ^2	$-\alpha$	i
σ^3	$-i\alpha$	i
τ	α	$-i$
$\sigma\tau$	$i\alpha$	$-i$
$\sigma^2\tau$	$-\alpha$	$-i$
$\sigma^3\tau$	$-i\alpha$	$-i$

5. From the table in Ex.4. we observe that

$$\tau(\sigma(\alpha)) = -i\alpha \qquad \text{and} \qquad \tau(\sigma(i)) = -i.$$

Hence $\tau\sigma = \sigma^3\tau = \sigma^{-1}\tau$, and this determines the multiplication completely:

$$(\sigma^i\tau^j)(\sigma^k\tau^l) = \sigma^{i+(-1)^j k}\tau^{j+l}$$

where the exponents at σ and τ are understood to be taken modulo 4 and 2, respectively. Indeed, if $j = 0$, the multiplication formula turns into

$$(\sigma^i)(\sigma^k\tau^l) = \sigma^{i+k}\tau^l,$$

which is obviously true, and if $j = 1$, the formula turns into

$$(\sigma^i\tau)(\sigma^k\tau^l) = \sigma^{i-k}\tau^{1+l}$$

which is true as well as the right hand side can be obtained from the left hand side by applying the earlier established rule $\tau\sigma = \sigma^{-1}\tau$ k times.

6. We know already from Ex.3. that σ has order 4 and τ has order 2. Hence σ^2 has order 2 and σ^3 has order 4. Of course, the identity has order 1. For the remaining three elements $\sigma^i\tau$ with $i = 1, 2, 3$ we have, using the formulae from Ex.5.,

$$(\sigma^i\tau)^2 = (\sigma^i\tau\sigma^i\tau = \sigma\sigma^{-i}\tau\tau = 1.$$

So they have order 2.

7. The cyclic subgroups of G are as follows:
$\langle\sigma\rangle = \langle\sigma^3\rangle$ (order 4),
$\langle\sigma^2\rangle, \langle\tau\rangle, \langle\tau\sigma\rangle, \langle\tau\sigma^2\rangle, \langle\tau\sigma^3\rangle$ (order 2),
$\langle 1\rangle$ (order 1).

8. Since G has order 8 the centre of G has order 2. Indeed, the centre is non-trivial since G is a finite 2-group, and it cannot have order 4 since the centre of a non-abelian group cannot have index 2. Finally, since σ^2 commutes with both generators σ and τ, it is a central element, so $\langle\sigma^2\rangle \leq Z(G)$, and since the centre has order 2 there must be equality.

9. The non-cyclic subgroups of G are:

$$\langle\sigma^2\rangle \times \langle\tau\rangle = \{1, \sigma^2, \tau, \sigma^2\tau\}$$

and

$$\langle\sigma^2\rangle \times \langle\tau\sigma\rangle = \{1, \sigma^2, \sigma\tau, \sigma^3\tau\}.$$

There cannot be any more non-cyclic proper subgroups: These must be generated by two elements of order 2. There are 5 elements of order 2 in G, hence we can form 10 pairs of elements of order 2, of these 10 pairs, 3 generate the first of our two non-cyclic subgroups of order, and another 3 generate the second. Each of the remaining 4 pairs, $\tau, \sigma\tau$; $\tau, \sigma^3\tau$; $\sigma\tau, \sigma^2\tau$ and $\sigma^3\tau, \sigma^2\tau$ generates all of G (as the group generated by them contains σ).

10. This is about the fixed fields of the three subgroups of order 4 in G. By the Fundamental Theorem, these subfields are of degree 2 over $\mathbb{Q}$. Since i is fixed by σ, $\sqrt{2} = \alpha^2$ is fixed by σ^2 and τ, $i\sqrt{2} = i\alpha^2$ is fixed by σ^2 and $\sigma\tau$, and all three intermediate fields $\mathbb{Q}(i)$,$\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(i\sqrt{2})$ have degree 2 over $\mathbb{Q}$, they are the fixed fields of the relevant three subgroups.

11. The only normal subgroup of order 2 in G is $\langle\sigma^2\rangle$. This subgroup is even central as we have seen in Ex.8. All other subgroups of order 2 are not normal as $\sigma^{-1}\sigma^i\tau\sigma = \sigma^{i-2}\tau \neq \sigma^i\tau$ for $i = 0, 1, 2, 3$.

12. This is about the fixed fields of subgroups of order 2 in G. By the Fundamental Theorem, these subfields are of degree 4 over $\mathbb{Q}$. Now, τ fixes α and $(\mathbb{Q}(\alpha) : \mathbb{Q}) = 4$. This proves (a). Next, σ^2 fixes both i and $\sqrt{2} = \alpha^2$, and $(\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q}) = 4$. This proves (b). Finally,

$$\sigma(\tau((1 + i)\alpha) = \sigma((1 - i)\alpha) = (1 - i)i\alpha = (1 + i)\alpha,$$

so $(1 + i)\alpha$ is fixed by $\sigma\tau$. Hence $\mathbb{Q}((1 + i)\alpha)$ is contained in $\Phi(\langle\sigma\tau\rangle)$. To be sure that we have actually equality, that is $\mathbb{Q}((1 + i)\alpha) = \Phi(\langle\sigma\tau\rangle)$, we need to verify that $(\mathbb{Q}((1 + i)\alpha) : \mathbb{Q}) = 4$. One way of seeing this is to observe that $(1 + i)\alpha)^2 = 2i\alpha^2$, and hence $\mathbb{Q}(i\alpha^2) \subseteq \mathbb{Q}((1 + i)\alpha)$. Now $(\mathbb{Q}(i\alpha^2) : \mathbb{Q}) = 2$ since $i\alpha = i\sqrt{2}$ has minimum polynomial $x^2 + 2$. But also $\mathbb{Q}(i\alpha^2) \neq \mathbb{Q}((1 + i)\alpha)$ since $i\sqrt{2}$ is fixed by σ^2 (see 10(c)), whereas $(1 + i)\alpha$ is not:

$$\sigma^2((1 + i)\alpha) = -(1 + i)\alpha.$$

So $(\mathbb{Q}((1 + i)\alpha) : \mathbb{Q}) = 4$ as required. Alternatively, observe that the elements $1, \alpha, \alpha^2, \alpha^3, i, i\alpha, i\alpha^2, i\alpha^3$ form a $\mathbb{Q}$-basis of E. Hence any element $\theta \in E$ has a unique expression as

$$\theta = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + b_0 i + b_1 i\alpha + b_2 i\alpha^2 + b_3 i\alpha^3$$

with $a_0, a_1, \ldots, b_3 \in \mathbb{Q}$. Applying $\sigma\tau$ gives

$$\sigma\tau(\theta) = a_0 + b_1\alpha - a_2\alpha^2 - b_3\alpha^3 - b_0 i + a_1 i\alpha + b_2 i\alpha^2 - a_3 i\alpha^3.$$

Consequently, θ is fixed by $\sigma\tau$ if and only if $a_1 = b_1$, $a_2 = 0$, and $-a_3 = b_3$. Hence $\Phi(\langle\sigma\tau\rangle)$ consists of all elements of the form

$$a_0 + a_1(1+i)\alpha + b_2 i\alpha^2 + a_3(1-i)\alpha^3.$$

Since $((1+i)\alpha)^2 = 2i\alpha^2$ and $((1+i)\alpha)^3 = 2i(1+i)\alpha^3 = -2(1-i)\alpha^3$, we have that $\Phi(\langle\sigma\tau\rangle) = \mathbb{Q}((1+i)\alpha)$.