MATH 2370, Homework 10

Kiumars Kaveh

Due date: Friday December 11, 2015

Problem 1: Suppose A and B are $n \times n$ real matrices which commute i.e. $AB = BA$. Prove that:

$$r(AB) \leq r(A)r(B).$$

(Hint: use Gelfand's formula that $r(A)$ is limit of $||A^k||^{1/k}$.)

Problem 2: Let $A = (a_{ij})$ be an $n \times n$ matrix (real or complex) with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that:

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j=1}^{n} |a_{ij}|^2.$$

Problem 3: Let A be a real $n \times n$ matrix that is orthogonally diagonalizable (i.e. there is an orthogonal matrix U such that $U^{-1}AU$ is diagonal).

(a) Show that $||U^{-1}AU|| = ||A||$.

(b) Prove that $r(A) = ||A||$.

Problem 4: Let A be an $n \times n$ matrix. Consider the series

$$\sum_{i=0}^{\infty} \frac{A^i}{i!}.$$

(a) Show that the above series is convergent with respect to the operator norm. The limit of the series is called the exponential of A and denoted by e^A.

(b) Show that if A is a symmetric matrix then e^A is also symmetric. Is it true that $||e^A|| = e^{||A||}$?