MATH 2370, Homework 2

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September 17, 2015

Due date: Friday September 25, 2015

Problem 1: Let K be a field and consider the vector space $X = K^n$. Recall from the lectures that the map $f : K^n \to (K^n)'$:

$$f : (a_1, \ldots, a_n) \mapsto \ell$$

where $\ell(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i$, gives an isomorphism between K^n and its dual space $(K^n)'$.

(1) Show that under the isomorphism f the standard basis goes to its dual basis. (Recall that the standard basis for K^n is the basis $\{e_1, \ldots, e_n\}$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (one in the i-th position).)

(2) Let $n = 2$. Consider the basis $B = \{(1, 1), (1, -1)\}$ for $X = K^2$. What basis for K^2 corresponds to the dual basis B' under the isomorphism f?

Problem 2: Let X be the (infinite dimensional) vector space of all sequences of real numbers:

$$X = \{s = (a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\}.$$

Also let Y be the vector space of all sequences of real numbers which are eventually 0:

$$Y = \{s = (a_1, a_2, \ldots) \mid \text{ There exists } N \text{ depending on } s \text{ such that } a_N = a_{N+1} = \cdots = 0\}.$$

(1) Show that Y is a vector space with a countable basis.

(2) Show that the dual space Y' can be identified with X. (One can show that X does not have a countable basis.)

(3) (Bonus) show that the natural linear map $Y \to Y''$ (explained in class) is not an isomorphism.

Problem 3: Consider the vector space P_n consisting of all real polynomials of degree $< n$. Fix k distinct points $t_1, \ldots, t_k$ in $\mathbb{R}$. Let $T : P_n \to \mathbb{R}^k$ be the linear map defined by:

$$T(p) = (p(t_1), \ldots, p(t_k)),$$

for all polynomials $p \in P_n$.

(1) Determine the dimensions of the null space and the image of T.

(2) Show that T is an isomorphism if and only if $k = n$. Conclude that if $a_1, \ldots, a_n$ are any given real numbers then there exists a unique polynomial p of degree $< n$ such that $p(t_i) = a_i$ for all $i = 1, \ldots, n$ (Lagrange's interpolation).

Problem 4: Let X be a finite dimensional vector space and let $T : X \to X$ be a linear mapping. Suppose the dimensions of the images of T and T^2 are the same. Prove that the intersection of the image R_T and the null space N_T is $\{0\}$. (Hint: you need to use $\dim(R_T) + \dim(N_T) = \dim(X)$.)