MATH 3650, Symplectic Geometry

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Due: November 8, 2013

All manifolds, functions, forms and vector fields are assumed to be C^∞, unless otherwise stated.

Problem 1: (Gradient-Hamiltonian vector field) Let M be a Kähler manifold with Kähler form ω and associated Riemannian metric g. Let $f : M \to \mathbb{C}$ be a holomorphic function on M. For each $z \in \mathbb{C}$ let M_z denote the fiber $f^{-1}(z)$. Suppose z is a regular value of f then we know that M_z is a complex submanifold of M and $\omega_z := \omega_{|M_z}$ is a Kähler form on X_z.

(a) Let $X_{\mathrm{Im}(f)}$ denote the Hamiltonian vector field of the imaginary part of f with respect to the symplectic form ω. Also let $\nabla(\mathrm{Re} f)$ denote the gradient vector field of the real part of f with respect to the Riemannian metric g. Show that

$$\nabla(\mathrm{Re} f) = -X_{\mathrm{Im} f}.$$

(Hint: Cauchy-Riemann relations.)

(b) Define the *gradient-Hamiltonian vector field* V_f by:

$$V_f := -\frac{\nabla(\mathrm{Re} f)}{|\nabla(\mathrm{Re} f)|^2} = \frac{X_{\mathrm{Im} f}}{|X_{\mathrm{Im} f}|^2}.$$

Prove that directional derivative (Lie derivative) of $\mathrm{Re} f$ in the direction of V_f is constantly equal to -1, and the directional derivative of $\mathrm{Im} f$ in the direction of V_f is constantly equal to 0.

(c) Let ϕ_t denote the flow of V. Show that, whenever defined, ϕ_t maps M_z to M_{z-t}. Moreover, show that ϕ_t respects the family of Kähler forms ω_z on M_z i.e., whenever defined $(\phi_t)^*(\omega_{z-t}) = \omega_z$.

(d) Let $M = \mathbb{C}^2$ (equipped with its standard Hermitian/Kähler structure) and let $f(x, y) = xy$. Compute V_f.

From DaSilva's book:
Homework 8 (Compatible Linear Structures) Problems 2, 3
Homework 12 (Fubini-Study Structure) Problems 5, 7 (use definition of F-S form given in class)
Homework 13 (Simple Pendulum) (c).
Homework 16 (Hermitian Matrices) Problems 1, 2, 5