MATH 2370, Homework 3

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As usual we follow the notation in P. Lax, Linear Algebra and Its Applications.

Do only 4 out of the 5 problems.

Problem 1: Let X be a finite dimensional vector space. Let $T, S \in \mathcal{L}(X, X)$ be linear maps. Show that

(1) $\dim(R_{ST}) \leq \dim(R_S)$.

(2) $\dim(N_{ST}) \leq \dim(N_S) + \dim(N_T)$.

Problem 2: Let X be a finite dimensional vector space and $W \subset X$ a subspace. Show that there is a projection map $P : X \to X$ such that the range of P is W (i.e. $P(X) = W$). (Recall that a linear mapping $P : X \to X$ is a projection if $P^2 = P$.)

Problem 3: Let Y, Z be subspaces of a finite dimensional vector space X. Show that:

(1) $(Y + Z)^\perp = Y^\perp \cap Z^\perp$.

(2) $(Y \cap Z)^\perp = Y^\perp + Z^\perp$.

Problem 4: Let X be the vector space of all real polynomials (in one variable t) of degree less than n. Let $c_1, \ldots, c_n$ be distinct real numbers and let $T : X \to \mathbb{R}^n$ be the linear map defined by:

$$T(p) = (p(c_1), \ldots, p(c_n)).$$

Consider the basis $\mathcal{B} = (1, t, t^2, \ldots, t^{n-1})$ for X and the standard basis $\mathcal{E} = (e_1, \ldots, e_n)$ for the vector space $\mathbb{R}^n$. Write the $n \times n$ matrix of the linear mapping T with respect to the bases $\mathcal{B}$ and $\mathcal{E}$.

Problem 5:

(1) Let X be a finite dimensional vector space. Let $T \in \mathcal{L}(X, X)$ be an invertible linear map. Show that T' is also invertible and $(T')^{-1} = (T^{-1})'$.

(2) Let $G = \mathrm{GL}(n, \mathbb{R})$ be the group of invertible $n \times n$ matrices. Show that the map $i : A \mapsto (A^{-1})^t$ is an automorphism of the group G and $i^2 = i$.