MATH 2370, Homework 6

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Problem 1: Let $p \in S_n$ be a permutation. To p we associate an $n \times n$ matrix P defined as follows: for every i, the i-th row of P is the standard basis (row) vector $e_{p(i)}$. Alternatively, for every j, the j-th column of P is the standard basis (column) vector $e_{p^{-1}(j)}$. In other words:

$$P_{ij} = \begin{cases} 1, & \text{if } j = p(i) \\ 0 & \text{otherwise.} \end{cases}$$

Show that if x is a column vector then Px is the vector obtained by permuting the components of x using the permutation p . Also show that if p_1, p_2 are permutations with corresponding matrices P_1, P_2 respectively, then the permutation matrix corresponding to p_1p_2 is the matrix P_1P_2. That is, the product of permutations correspond to product of matrices. (In fact, this problem was discussed in class.)

Problem 2: Let A be a nilpotent matrix, i.e. there is $m > 0$ such that $A^m = 0$. Show that A is not diagonalizable unless $A = 0$. Recall that a matrix is diagonalizable if it is similar to a diagonal matrix.

Problem 3: Find the characteristic polynomial as well as the minimal polynomial for the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

Problem 4: Let $n \geq 2$ and let A be an $n \times n$ matrix with $\text{rank}(A) = 1$ (that is, the span of columns is one dimensional, or in other words all the columns are scalar multiples of one column).

(a) Show that there exist two (column) vectors a and b such that $A = ab^T$.

(b) Show that the minimal polynomial of A is $\lambda^2 - (a^T b)\lambda$.