MATH 2370, Homework 8

Kiumars Kaveh

Due date: Monday November 16, 2015

Problem 1: Let A be an $n \times n$ complex matrix. A vector $v \in \mathbb{C}^n$ is called a *cyclic vector for* A if $\{v, Av, A^2v, \ldots, A^{n-1}v\}$ span $\mathbb{C}^n$. Suppose A has a cyclic vector. (Note: not every matrix has a cyclic vector!)

(a) Show that the minimal polynomial of A and the characteristic polynomial of A coincide.

(b) Prove that each eigenvalue λ of A corresponds to one Jordan block.

Problem 2: As above let A be an $n \times n$ matrix with a cyclic vector v.

(a) Show that any vector $w \in \mathbb{C}^n$ is of the form $w = f(A)v$ where $f(t) \in \mathbb{C}[t]$ is a polynomial.

(b) (Cyclic vector theorem) Show that if B is another matrix which commutes with A then $B = f(A)$ for some polynomial $f(t) \in \mathbb{C}[t]$. Hint: By Part (a) we can find a polynomial f such that $Bv = f(A)v$. Show that $f(A) = B$.

Problem 3: Let λ be an eigenvalue of an $n \times n$ matrix A. Suppose that $\dim(N_1(\lambda)) = 2$, $\dim(N_2(\lambda)) = 4$ and $\dim(N_3(\lambda)) = \dim(N_4(\lambda) = 5$. Find the Jordan blocks of A corresponding to λ. (Recall that $N_k(\lambda)$ is the null space of $(A - \lambda I)^k$).

Problem 4: Let P be the linear space of polynomials with real coefficients equipped with the scalar product

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

(a) Using Gram-Schmidt process to generate an orthonormal basis of the span of vectors $\{1, x^2\}$.

(b) Find the projection of polynomial x on the span of vectors $\{1, x^2\}$.