MATH 2370, Homework 9

Kiumars Kaveh

Due date: Monday November 30, 2015

Problem 1: Let V be a finite dimensional vector space over a field K and let $\dim(V) = n$. Let $B : V \times V \to K$ be a symmetric bilinear form on V. Let $\{b_1, \ldots, b_n\}$ be a basis for V. Define the matrix M by $M_{ij} = B(b_i, b_j)$.

(a) Prove that for any $v, w \in V$ with $v = \sum_{i=1}^n x_i b_i$ and $w = \sum_{j=1}^n y_j b_j$ we have:
$$B(v, w) = x^T M y,$$
where $x = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$.

(b) Show that B is non-degenerate if and only if M is an invertible matrix.

(c) Define the symmetric bilinear form B on $\mathbb{R}^2$ by:
$$B(x, y) = x_1 x_2 - y_1 y_2,$$
where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Show that B is not a scalar product.

Problem 2: Suppose B is a non-degenerate symmetric bilinear form on a finite dimensional vector space V. Let W be a subspace of V and as usual let
$$W^\perp = \{x \in V \mid B(x, w) = 0,\ \forall w \in W\}.$$
Prove that $\dim(W) + \dim(W^\perp) = \dim(V)$. Note that in class we proved this for an scalar product $(\cdot, \cdot)$. Hint: for each $v \in V$ consider the linear function ℓ_v on W defined by $\ell_v(w) = B(v, w)$. This gives a linear map from V to the dual space W'. Show that this map is onto and its null space (kernel) is exactly $W^\perp$.

Problem 3: Let $V \subset \mathbb{R}[t]$ be the vector space of polynomials of degree at most 2. Equip V with the scalar product:

$$(f, g) = \int_0^1 f(t)g(t)dt.$$

Consider the subspace W consisting polynomials of degree at most 1. Find polynomial $h(t) \in W$ which has minimum distance to $f(t) = t^2$.

Problem 4: Let A and B be $n \times n$ real orthogonal matrices such that $\det(A) = 1$ and $\det(B) = -1$. Show that there is a unit vector $x \in \mathbb{R}^n$ such that $Ax = -Bx$. Hint: multiply both sides of $Ax = -Bx$ by B^{-1}. Thus replacing $B^{-1}A$ by A the problem is reduced to the following: if A is an orthogonal matrix with determinant -1 then A has -1 as an eigenvalue. To do this show that the characteristic polynomial p_A has a negative root which then must be -1.

Problem 5: Find the operator norms of the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

with respect to the standard Euclidean structures on $\mathbb{R}^2$? Hint: for A diagonalize it. For B note that it is orthogonal after dividing by a scalar.

Problem 6: (Bonus) Suppose A is an $n \times n$ real matrix such that $A^m = I$ for some $m > 0$. Consider the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ defined using A by:

$$\langle v, w \rangle = \frac{1}{m} \sum_{i=0}^{m-1} (A^i v, A^i w),$$

where $(\cdot, \cdot)$ denotes the standard scalar product on $\mathbb{R}^n$. Prove that A is an orthogonal matrix with respect to the scalar product $\langle \cdot, \cdot \rangle$.