**Department of Mathematics**

# University of Pittsburgh

# MATH 2500

## **Midterm**, Spring 2018

Instructor: Kiumars Kaveh

**Last Name:** **Student Number:**

**First Name:**

TIME ALLOWED: 50 MINUTES. TOTAL: 60
NO AIDS ALLOWED. WRITE SOLUTIONS ON THE SPACE PROVIDED.

| Question | Mark |
|---|---|
| 1 | /15 |
| 2 | /5 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| 6 | /10 |
| 7 (bonus) | /5 |
| TOTAL | /60 |

**1(a).**[10 points] Give the definitions of the following: (i) An action of a group $G$ on a set $X$. (ii) A solvable group. (iii) A composition series for a group $G$.

**(b).**[5 points] State the Jordan-Hölder theorem, and the 1st isomorphism theorem.

**2.**[5 points] Prove the orbit-stabilizer theorem: let $G$ be a group acting on a set $X$. Let $H = \text{stab}_G(x)$ be the stabilizer subgroup of $x \in X$. Then there is a one-to-one correspondence between the orbit $G{\cdot}x$ and the coset space $G/H$.

**3.**[10 points] Show that $A_4$ is not a simple group. Does $A_4$ have a subgroup of index 2? Justify your answer.

**4.**[10 points] Let $X$ be a finite set and suppose $G$ is an abelian subgroup of the symmetric group $S_X$ that acts transitively on $X$. Show that for all $e \neq g \in G$ and all $x \in X$ we have $g \cdot x \neq x$. Deduce that $|G| = |X|$. (Hint: recall that if $y = g \cdot x$ then $g\,\mathrm{stab}(x)\,g^{-1} = \mathrm{stab}(y)$.)

**5.**[10 points] Find all the conjugacy classes in $S_4$.

**6.**[10 points] Let $G$ be a finite group with $|G| = n$. Let $k$ be an integer relatively prime to $n$. Show that the map $x \mapsto x^k$, $\forall x \in G$, is surjective. (Hint: first prove the statement for when $G$ is a cyclic group, you can use theorems proved in class.)

**7.**[5 points] (bonus) Let $G$ be a group. Let $N$ be the subgroup of $G$ generated by all the elements of the form $xyx^{-1}y^{-1}$, $\forall x, y \in G$.

(a) Show that if $\phi : G \to G$ is any automorphism of $G$ then $\phi(N) = N$. Conclude that $N$ is a normal subgroup of $G$.

(b) Show that $G/N$ is abelian ($N$ is usually called the *commutator subgroup* of $G$).