Some practice problems for midterm 1

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Problem: Which one of the following is a cyclic group? Give a generator for the group if it is cyclic, and if not, argue why (i.e. can not be generated by one element).

(a) $(\mathbb{Q} \setminus \{0\}, \times)$

Solution: $(\mathbb{Q}\backslash 0, \times)$ is not cyclic. Let's prove by contradiction. Suppose it is generated by a, then $a^n = -1$ for some $n \in \mathbb{Z}$. This implies $a = -1$. But $< -1 >= \{-1, 1\} \neq (\mathbb{Q}\backslash 0, \times)$.

(b) Symmetry group of a square

Solution: D_4 is not a cyclic group. $|D_4| = 8$, but the order of every element is less or equal to 4.

(c) $(\mathbb{Z}_{10}^*, \times)$.

Solution: $\mathbb{Z}_{10}^* = \{1, 3, 7, 9\}$ is cyclic. 3 is a generator.

Problem: Give an isomorphism between $(\mathbb{Z}_8^*, \times)$ and the Klein group V.

Solution: $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$, $V_4 = \{1, a, b, c\}$. Drawing the multiplication table for $\mathbb{Z}_8^*$ one checks that $\phi(1) = 1, \phi(3) = a$, $\phi(5) = b$, $\phi(7) = c$ gives an isomorphism.

Problem: Verify that $(\mathbb{Z}_9^*, \times)$ is cyclic and find all its subgroups.

Solution: $\mathbb{Z}_9^* = \{1, 2, 4, 5, 7, 8\}$ is cyclic, since $\mathbb{Z}_9^* = \langle 2\rangle$. Subgroups: $\langle 1\rangle, \{1, 8\}, \{1, 4, 7\}, \mathbb{Z}_9^*$.

Problem: Find all the elements of order 10 in $(\mathbb{Z}_{30}, +)$.

Solution: We know that the order of a is equal to $30/d$ where $d = (a, 30)$ (Theorem 6.14). Thus a is of order 10 if and only if $(a, 30) = 3$. Therefore, $a = 3, 9, 21, 27$.

Problem: Let $G = A_4$ the alternating group in 4 elements. Let $\sigma = (134) \in A_4$ and $H = \langle\sigma\rangle$. Find all the cosets of H in A_4.

Solution: σ has order 3 because it is a 3-cycle and hence $H = \{e, \sigma, \sigma^2\}$. A_4 has $4!/2 = 12$ elements. Hence $[A_4 : H] = 12/3 = 4$, i.e. there are 4 cosets. H is always one of the cosets. We need to find 3 other cosets. In fact, one can verify that the four 3-cycles: $\sigma_1 = \sigma$, $\sigma_2 = (123)$, $\sigma_3 = (124)$ and $\sigma_4 = (234)$ give representatives for the four different cosets, that is, for any $i \neq j$ we have $\sigma_i \sigma_j^{-1} \notin H$. Thus the four cosets are: $\sigma_1 H$, $\sigma_2 H$, $\sigma_3 H$ and $\sigma_4 H$.

Problem: How many non-isomorphic abelian groups of order 180 are there? List all of them.

Solution: $180 = 2^2 3^2 5$.

$$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5.$$

Problem: Does the symmetry group of a square (i.e. D_4) has a subgroup isomorphic to V? how about a subgroup isomorphic to $\mathbb{Z}_4$?

Solution: Consider the group consisting of vertical and horizontal reflections, rotation by π and identity. This group is isomorphic to V. The group generated by rotation by $\pi/2$ is isomorphic to $\mathbb{Z}_4$.

Problem: Recall that GL$(2, \mathbb{R})$ denotes the group of all 2×2 invertible matrices (with real entries) and matrix multiplication. Is the set of all 2×2 invertible symmetric matrices (with real entries) a subgroup of GL$(2, \mathbb{R})$? A matrix A is symmetric if it is symmetric with respect to its diagonal, in other words, $A = A^t$.

Solution: No. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} = \begin{pmatrix} aa' + bb' & ab' + bc' \\ ba' + cb' & bb' + cc' \end{pmatrix}$. In general $ab' + bc' \neq ba' + cb'$. So not closed under matrix multiplication.

Problem: Show that a group G is abelian if and only if we have

$$(ab)^{-1} = a^{-1} b^{-1} \quad \forall a, b \in G.$$

Solution: Suppose G is abelian that is $ab = ba$, then $(ab)^{-1} = (ba)^{-1} = a^{-1}b^{-1}$. Conversely, suppose $(ab)^{-1} = a^{-1}b^{-1}$. We know $(ab)^{-1} = b^{-1}a^{-1}$. So $a^{-1}b^{-1} = b^{-1}a^{-1}$ $\forall a, b \in G$. Replacing a with a^{-1} and b with b^{-1} we conclude that G is abelian.

Problem: Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 2 & 3 & 6 & 5 & 1 \end{pmatrix},$$

and $\tau = (21)(23)(45)(57)$. Write σ and τ as product of disjoint cycles. Determine if they are odd or even.

Solution: $\sigma = (17)(243)(56)$. We multiply out $(12)(23)(45)(57)$ to write τ in the array form:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 7 & 6 & 4 \end{pmatrix}.$$

Then $\tau = (123)(457)$. Since (243) is even σ is also even. (123) and (457) are even and hence τ is also even.

Problem: Let G be the dihedral group D_5. Elements of G are rotations and reflections. Let r be the counter-clockwise rotation by $2\pi/5$ radian and s be the reflection with respect to the x-axis. Recall that we have the relations: $rs = sr^{-1}, s^2 = 1$.

(i) **Write the following elements of G in terms of r and s: (1) clockwise rotation by $4\pi/5$ radian. (2) Reflection with respect to the line l which passes through the origin and makes an angle of $2\pi/5$ with the x-axis.**

1) $r^{-2} = r^3$; 2) $r^2 s$ (Note that we first apply s then r^2.)

(ii) **Simplify the following element and decide if it is a rotation or reflection: $rrsr^{-1}sr^{-1}sr^3sr^2$.**

Using the relations $rs = sr^{-1}, s^2 = 1$ we have $rrsr^{-1}sr^{-1}sr^3sr^2 = r^3s^2r^{-4}s^2r^2 = r$, therefore, it is a rotation.

Problem: Let $\phi : G \to G'$ be an onto homomorphism (surjective). Show that if G is abelian then G' is abelian. Also show that if G is cyclic then G' is also cyclic.

Proof: Suppose G is abelian. We want to show that G' is abelian. Take $x', y' \in G'$. Since ϕ is onto we can find $x, y \in G$ with $\phi(x) = x'$, $\phi(y) = y'$. Now, $x'y' = \phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x) = y'x'$ as required.

Next let us show that if G is cyclic then G' is cyclic. Let a be a generator for G, we show that $a' = \phi(a)$ is a generator for G'. Take $x' \in G'$. Then there is $x \in G$ with $\phi(x) = x'$. Since a is a generator for G we have $x = a^k$ for some $k \in \mathbb{Z}$. Then $x' = \phi(x) = \phi(a^k) = (\phi(a))^k = a'^k$ which proves that a' is a generator for G'.

Problem: Let $\phi : G \to G'$ be an onto homomorphism. Is it true that if G' is cyclic then G is also cyclic?

Solution: No. Example consider the map $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ given by $(x, y) \mapsto x$. One verifies that ϕ is a homomorphism. Now $\mathbb{Z}_2$ is cyclic while $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not.

Problem: What is the symmetry group of the following shapes: (1) Yin and Yang Buddhist symbol (ignoring the black-white color). (2) Star of David. (3) Red Cross.

Solution: 1) $\mathbb{Z}_2$, 2)D_6, 3)D_4.

Problem: Let G be any group and let a be any element of G. Let $\phi : \mathbb{Z} \to G$ be defined by $\phi(n) = a^n$. Show that ϕ is a homomorphism. Describe the image of ϕ and the kernel of ϕ.

Solution: We have $\phi(n+m) = a^{n+m} = a^n a^m = \phi(n)\phi(m)$ thus ϕ is a homomorphism. The image of ϕ is $\{a^n \mid a \in \mathbb{Z}\}$ which is the subgroup $\langle a \rangle$ generated by a. If $k = \text{ord}(a)$ then the kernel of ϕ is the subgroup $k\mathbb{Z}$. If a has infinite order then $\ker(\phi) = \{0\}$.

Problem: Let $\phi : G \to G'$ be an onto homomorphism where G, G' are finite groups. Show that $|G'|$ divides $|G|$.

Proof: Let H be the kernel of ϕ. By a theorem proved in class we know that $\phi(x) = \phi(y)$, for $x, y \in G$, if and only if $xH = yH$. Thus, as ϕ is onto, $|G'|$ is equal to the number of cosets of H, namely the index of H in G. Hence $|G| = |H||G'|$ which proves the claim.

Problem: Find the order of the element $(3, 10)$ in $\mathbb{Z}_5 \times \mathbb{Z}_{18}$.

Solution: The order of 3 in $\mathbb{Z}_5$ is 5 since $(3, 5) = 1$. Also $(10, 18) = 2$ and thus the order of 10 in $\mathbb{Z}_{18}$ is $18/2 = 9$ (Theorem 6.14). Now the smallest number k such that $k3 \equiv 0 \bmod 5$ and $k10 \equiv 0 \bmod 18$ is the least common multiple of 5 and 9 which is 45.

Problem: Give an example of each of the following. Justify your example briefly.

(a) A group G with $|G| = 6$ and two subgroups H and K with $|H| = 2$ and $|K| = 3$ such that G is not isomorphic to $H \times K$.

Solution: $G = S_3$, $H = \langle(12)\rangle$ and $K = \langle(123)\rangle$. Then S_3 is not isomorphic to $H \times K$ because H and K are cyclic and hence abelian, and thus $H \times K$ is also abelian, while S_3 is not abelian.

(b) A subgroup of $\mathbb{Z}_{10} \times \mathbb{Z}_{20}$ isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_2$.

Solution: Let $H < \mathbb{Z}_{10}$ be the subgroup generated by 2, and $K < \mathbb{Z}_{20}$ be the subgroup generated by 10. Then $H \cong \mathbb{Z}_5$, $K \cong \mathbb{Z}_2$ and $H \times K \cong \mathbb{Z}_5 \times \mathbb{Z}_2$.

(c) A subgroup of S_4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution: Subgroup generated by $\sigma = (12)$ and $\tau = (34)$.