In fact, I tried it out with two integers, each having more than one thousand digits. It found the product in about one second." Ever the skeptic, Zori said "You mean you carefully typed in two integers of that size?" Xing quickly replied "Of course not. I just copied and pasted the data from one source to another." Yolanda said "What a neat trick that is. Really cuts down the chance of an error."

Dave said "What about factoring? Can your netbook with its fancy software for strings factor big integers?" Xing said that he would try some sample problems and report back. Carlos said "Factoring an integer with several hundred digits is likely to be very challenging, not only for a netbook, but also for a super computer. For example, suppose the given integer was either a prime or the product of two large primes. Detecting which of these two statements holds could be very difficult."

Undeterred, Dave continued "What about exponentiation? Can your software calculate a^b when a and b are large integers?" Xing said "That shouldn't be a problem. After all, a^b is just multiplying a times itself a total of b times, and if you can do multiplication quickly, that's just a loop." Yolanda said that the way Xing was describing things, he was actually talking about a program with nested loops so it might take a long time for such a program to halt. Carlos was quiet but he thought there might be ways to speed up such computations.

By this time, Alice reinserted herself into the conversation "Hey guys. While you were talking, I was looking for big integer topics on the web and found this problem. Is 838200020310007224300 a Catalan number? How would you answer this? Do you have to use special software?"

Zori was not happy. She gloomily envisioned a future job hunt in which she was compelled to use big integer arithmetic as a job skill. Arrgghh.

2.9. Exercises

1. The Hawaiian alphabet consists of 12 letters. How many six-character strings can be made using the Hawaiian alphabet?

2. How many $2n$-digit positive integers can be formed if the digits in odd positions (counting the rightmost digit as position 1) must be odd and the digits in even positions must be even and positive?

3. Matt is designing a website authentication system. He knows passwords are most secure if they contain letters, numbers, and symbols. However, he doesn't quite understand that this additional security is defeated if he specifies in which positions each character type appears. He decides that valid passwords for his system will begin with three letters (uppercase and lowercase both allowed), followed by two digits, followed by one of 10 symbols, followed by two uppercase letters, followed by a digit, followed by one of 10 symbols. How many different

passwords are there for his website system? How does this compare to the total number of strings of length 10 made from the alphabet of all uppercase and lowercase English letters, decimal digits, and 10 symbols?

4. How many ternary strings of length $2n$ are there in which the zeroes appear only in odd-numbered positions?

5. Suppose we are making license plates of the form $l_1l_2l_3 - d_1d_2d_3$ where l_1, l_2, l_3 are capital letters in the English alphabet and d_1, d_2, d_3 are decimal digits (i.e., in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) subject to the restriction that at least one digit is nonzero and at least one letter is K. How many license plates can we make?

6. Mrs. Steffen's third grade class has 30 students in it. The students are divided into three groups (numbered 1, 2, and 3), each having 10 students.
   a) The students in group 1 earned 10 extra minutes of recess by winning a class competition. Before going out for their extra recess time, they form a single file line. In how many ways can they line up?
   b) When all 30 students come in from recess together, they again form a single file line. However, this time the students are arranged so that the first student is from group 1, the second from group 2, the third from group 3, and from there on, the students continue to alternate by group in this order. In how many ways can they line up to come in from recess?

7. How many strings of the form $l_1l_2d_1d_2d_3l_3l_4d_4l_5l_6$ are there where
   - for $1 \leq i \leq 6$, l_i is an uppercase letter in the English alphabet;
   - for $1 \leq i \leq 4$, d_i is a decimal digit;
   - l_2 is not a vowel (i.e., $l_2 \notin \{\text{A,E,I,O,U}\}$); and
   - the digits d_1, d_2, and d_3 are distinct (i.e., $d_1 \neq d_2 \neq d_3 \neq d_1$).

8. In this exercise, we consider strings made from uppercase letters in the English alphabet and decimal digits. How many strings of length 10 can be constructed in each of the following scenarios?
   a) The first and last characters of the string are letters.
   b) The first character is a vowel, the second character is a consonant, and the last character is a digit.
   c) Vowels (not necessarily distinct) appear in the third, sixth, and eight positions and no other positions.
   d) Vowels (not necessarily distinct) appear in exactly two positions.
   e) Precisely four characters in the string are digits and no digit appears more than one time.

9. A database uses 20-character strings as record identifiers. The valid characters in these strings are upper-case letters in the English alphabet and decimal digits. (Recall there are 26 letters in the English alphabet and 10 decimal digits.) How many valid record identifiers are possible if a valid record identifier must meet *all* of the following criteria:
   - Letter(s) from the set $\{A, E, I, O, U\}$ occur in *exactly* three positions of the string.
   - The last three characters in the string are *distinct* decimal digits that do not appear elsewhere in the string.
   - The remaining characters of the string may be filled with any of the remaining letters or decimal digits.

10. Let X be the set of the 26 lowercase English letters and 10 decimal digits. How many X-strings of length 15 satisfy *all* of the following properties (at the same time)?
    - The first and last symbols of the string are distinct digits (which may appear elsewhere in the string).
    - Precisely four of the symbols in the string are the letter $'t'$.
    - Precisely three characters in the string are elements of the set $V = \{a, e, i, o, u\}$ and these characters are all distinct.

11. A donut shop sells 12 types of donuts. A manager wants to buy six donuts, one each for himself and his five employees.
    a) Suppose that he does this by selecting a specific type of donut for each person. (He can select the same type of donut for more than one person.) In how many ways can he do this?
    b) How many ways could he select the donuts if he wants to ensure that he chooses a different type of donut for each person?
    c) Suppose instead that he wishes to select one donut of each of six *different* types and place them in the breakroom. In how many ways can he do this? (The order of the donuts in the box is irrelevant.)

12. The sport of korfball is played by teams of eight players. Each team has four men and four women on it. Halliday High School has seven men and 11 women interested in playing korfball. In how many ways can they form a korfball team from their 18 interested students?

13. Twenty students compete in a programming competition in which the top four students are recognized with trophies for first, second, third, and fourth places.
    a) How many different outcomes are there for the top four places?

b) At the last minute, the judges decide that they will award honorable mention certificates to four individuals who did not receive trophies. In how many ways can the honorable mention recipients be selected (after the top four places have been determined)? How many total outcomes (trophies plus certificates) are there then?

14. An ice cream shop has a special on banana splits, and Xing is taking advantage of it. He's astounded at all the options he has in constructing his banana split:

    - He must choose three different flavors of ice cream to place in the asymmetric bowl the banana split is served in. The shop has 20 flavors of ice cream available.
    - Each scoop of ice cream must be topped by a sauce, chosen from six different options. Xing is free to put the same type of sauce on more than one scoop of ice cream.
    - There are 10 sprinkled toppings available, and he must choose three of them to have sprinkled over the entire banana split.

    a) How many different ways are there for Xing to construct a banana split at this ice cream shop?

    b) Suppose that instead of requiring that Xing choose exactly three sprinkled toppings, he is allowed to choose between zero and three sprinkled toppings. In this scenario, how many different ways are there for him to construct a banana split?

15. Suppose that a teacher wishes to distribute 25 identical pencils to Ahmed, Barbara, Casper, and Dieter such that Ahmed and Dieter receive at least one pencil each, Casper receives no more than five pencils, and Barbara receives at least four pencils. In how many ways can such a distribution be made?

16. How many integer-valued solutions are there to each of the following equations and inequalities?

    a) $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i > 0$

    b) $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$

    c) $x_1 + x_2 + x_3 + x_4 + x_5 \leq 63$, all $x_i \geq 0$

    d) $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$, $x_2 \geq 10$

    e) $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$, $x_2 \leq 9$

17. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 132$$

provided that $x_1 > 0$, and $x_2, x_3, x_4 \geq 0$? What if we add the restriction that $x_4 < 17$?

18. How many integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 782$$

provided that $x_1, x_2 > 0$, $x_3 \geq 0$, and $x_4, x_5 \geq 10$?

19. A teacher has 450 identical pieces of candy. He wants to distribute them to his class of 65 students, although he is willing to take some leftover candy home. (He does not insist on taking any candy home, however.) The student who won a contest in the last class is to receive at least 10 pieces of candy as a reward. Of the remaining students, 34 of them insist on receiving at least one piece of candy, while the remaining 30 students are willing to receive no candy.

    a) In how many ways can he distribute the candy?

    b) In how many ways can he distribute the candy if, in addition to the conditions above, one of his students is diabetic and can receive at most 7 pieces of candy? (This student is one of the 34 who insist on receiving at least one piece of candy.)

20. Give a combinatorial argument to prove the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Hint: Think of choosing a team with a captain.

21. Let m and w be positive integers. Give a combinatorial argument to prove that for integers $k \geq 0$,

$$\sum_{j=0}^{k} \binom{m}{j}\binom{w}{k-j} = \binom{m+w}{k}.$$

22. How many lattice paths are there from $(0,0)$ to $(10,12)$?

23. How many lattice paths are there from $(3,5)$ to $(10,12)$?

24. How many lattice paths are there from $(0,0)$ to $(10,12)$ that pass through $(3,5)$?

25. How many lattice paths from $(0,0)$ to $(17,12)$ are there that pass through $(7,6)$ and $(12,9)$?

26. How many lattice paths from $(0,0)$ to $(14,73)$ are there that do *not* pass through $(6,37)$?

27. A small-town bank robber is driving his getaway car from the bank he just robbed to his hideout. The bank is at the intersection of 1st Street and 1st Avenue. He needs to return to his hideout at the intersection of 7th Street and 5th Avenue. However, one of his lookouts has reported that the town's one police officer is parked at the intersection of 4th Street and 4th Avenue. Assuming that the bank robber does not want to get arrested and drives only on streets and avenues, in how many ways can he safely return to his hideout? (Streets and avenues are uniformly spaced and numbered consecutively in this small town.)

28. The setting for this problem is the fictional town of Mascotville, which is laid out as a grid. Mascots are allowed to travel only on the streets, and not "as the yellow jacket flies." Buzz, the Georgia Tech mascot, wants to go visit his friend Thundar, the North Dakota State University mascot, who lives 6 blocks east and 7 blocks north of Buzz's hive. However, Uga VIII has recently moved into the doghouse 2 blocks east and 3 blocks north of Buzz's hive and already has a restraining order against Buzz. There's also a pair of tigers (mother and cub) from Clemson who live 1 block east and 2 blocks north of Uga VIII, and they're known for setting traps for Buzz. Buzz wants to travel from his hive to Thundar's pen every day without encountering Uga VIII or The Tiger and The Tiger Cub. However, he wants to avoid the boredom caused by using a route he's used in the past. What is the largest number of consecutive days on which Buzz can make the trip to visit Thundar without reusing a route (you may assume the routes taken by Buzz only go east and north)?

29. Determine the coefficient on $x^{15}y^{120}z^{25}$ in $(2x + 3y^2 + z)^{100}$.

30. Determine the coefficient on $x^{12}y^{24}$ in $(x^3 + 2xy^2 + y + 3)^{18}$. (Be careful, as x and y now appear in multiple terms!)

31. For each word below, determine the number of rearrangements of the word in which all letters must be used.
    a) OVERNUMEROUSNESSES
    b) OPHTHALMOOTORHINOLARYNGOLOGY
    c) HONORIFICABILITUDINITATIBUS (the longest word in the English language consisting strictly of alternating consonants and vowels[1])

32. How many ways are there to paint a set of 27 elements such that 7 are painted white, 6 are painted old gold, 2 are painted blue, 7 are painted yellow, 5 are painted green, and 0 of are painted red?

33. There are many useful sets that are enumerated by the Catalan numbers. (Volume two of R.P. Stanley's *Enumerative Combinatorics* contains a famous (or perhaps

[1] http://www.rinkworks.com/words/oddities.shtml

3.11. Exercises

For questions asking you find a recursive formula, be sure to give enough initial values to get the recursion going.

1. A database uses record identifiers that are alphanumeric strings in which the 10 decimal digits and 26 upper-case letters are valid symbols. The criteria that define a valid record identifier are recursive. A valid record identifier of length $n \geq 2$ can be constructed in the following ways:
   - beginning with any upper-case letter other than D and followed by any valid record identifier of length $n-1$;
   - beginning with $1C$, $2K$, or $7J$ and followed by any valid record identifier of length $n-2$; or
   - beginning with D and followed by any string of $n-1$ decimal digits.

   Let $r(n)$ denote the number of valid record identifiers of length n. We take $r(0) = 1$ and note that $r(1) = 26$. Find a recursion for $r(n)$ when $n \geq 2$ and use it to compute $r(5)$.

2. Consider a $1 \times n$ checkerboard. The squares of the checkerboard are to be painted white and gold, but no two consecutive squares may both be painted white. Let $p(n)$ denote the number of ways to to paint the checkerboard subject to this rule. Find a recursive formula for $p(n)$ valid for $n \geq 3$.

3. Give a recursion for the number $g(n)$ of ternary strings of length n that do not contain 102 as a substring.

4. A $2 \times n$ checkerboard is to be tiled using two types of tiles. The first tile is a 1×1 square tile. The second tile is called an L-tile and is formed by removing the upper-right 1×1 square from a 2×2 tile. The L-tiles can be used in any of the four ways they can be rotated. (That is, the "missing square" can be in any of four positions.) Let $t(n)$ denote the number of tilings of the $2 \times n$ checkerboard using 1×1 tiles and L-tiles. Find a recursive formula for $t(n)$ and use it to determine $t(7)$.

5. Let S be the set of strings on the alphabet $\{0, 1, 2, 3\}$ that do not contain 12 or 20 as a substring. Give a recursion for the number $h(n)$ of strings in S of length n. *Hint*: Check your recursion by manually computing $h(1)$, $h(2)$, $h(3)$, and $h(4)$.

6. Find $d = \gcd(5544, 910)$ as well as integers a and b such that $5544a + 910b = d$.

7. Find $\gcd(827, 249)$ as well as integers a and b such that $827a + 249b = 6$.

8. Let a, b, m, and n be integers and suppose that $am + bn = 36$. What can you say about $\gcd(m, n)$?

5.8. Discussion

Over coffee, today's conversation was enthusiastic and heated at times. Zori got things off with a blast "I don't think graphs are of any use at all. . . " but she wasn't even able to finish the sentence before Yolanda uncharacteristically interrupted her with "You're off base on this one. I see lots of ways graphs can be used to model real world problems. The professor actually showed us examples back in our first class. But now that we're talking in more depth about graphs, things are even clearer." Bob added "These eulerian and hamiltonian cycle problems are certain to have applications in network routing problems." Xing reinforced Bob with "Absolutely. There are important questions in network integrity and information exchange that are very much the same as these basic problems." Alice piled on "Even the notion of chromatic number clearly has practical applications." By this time, Zori realized her position was indefensible but she was reluctant to admit it. She offered only a "Whatever."

Things quieted down a bit and Dave said "Finding a hamiltonian cycle can't be all that hard, if someone guarantees that there is one. This extra information must be of value in the search." Xing added "Maybe so. It seems natural that it should be easier to find something if you know it's there." Alice asked "Does the same thing hold for chromatic number?" Bob didn't understand her question "Huh?" Alice continued, this time being careful not to even look Bob's way "I mean if someone tells you that a graph is 3-colorable, does that help you to find a coloring using only three colors?" Dave said "Seems reasonable to me."

After a brief pause, Carlos offered "I don't think this extra knowledge is of any help. I think these problems are pretty hard, regardless." They went back and forth for a while, but in the end, the only thing that was completely clear is that graphs and their properties had captured their attention, at least for now.

5.9. Exercises

1. The questions in this exercise pertain to the graph **G** shown in Figure 5.28.
   a) What is the degree of vertex 8?
   b) What is the degree of vertex 10?
   c) How many vertices of degree 2 are there in **G**? List them.
   d) Find a cycle of length 8 in **G**.
   e) What is the length of a shortest path from 3 to 4?
   f) What is the length of a shortest path from 8 to 7?
   g) Find a path of length 5 from vertex 4 to vertex 6.
2. Draw a graph with 8 vertices, all of odd degree, that does not contain a path of length 3 or explain why such a graph does not exist.

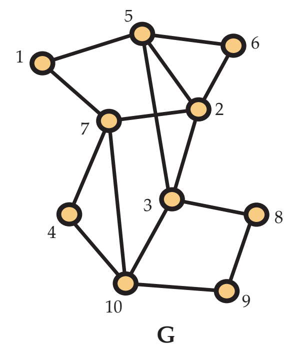


Figure 5.28.: A GRAPH

3. Draw a graph with 6 vertices having degrees 5, 4, 4, 2, 1, and 1 or explain why such a graph does not exist.

4. For the next Olympic Winter Games, the organizers wish to expand the number of teams competing in curling. They wish to have 14 teams enter, divided into two pools of seven teams each. Right now, they're thinking of requiring that in preliminary play each team will play seven games against distinct opponents. Five of the opponents will come from their own pool and two of the opponents will come from the other pool. They're having trouble setting up such a schedule, so they've come to you. By using an appropriate graph-theoretic model, either argue that they cannot use their current plan or devise a way for them to do so.

5. For this exercise, consider the graph **G** in Figure 5.29.
   a) Let $V_1 = \{g, j, c, h, e, f\}$ and $E_1 = \{ge, jg, ch, ef\}$. Is (V_1, E_1) a subgraph of **G**?
   b) Let $V_2 = \{g, j, c, h, e, f\}$ and $E_2 = \{ge, jg, ch, ef, cj\}$. Is (V_2, E_2) a subgraph of **G**?
   c) Let $V_3 = \{a, d, c, h, b\}$ and $E_3 = \{ch, ac, ad, bc\}$. Is (V_3, E_3) an induced subgraph of **G**?
   d) Draw the subgraph of **G** induced by $\{g, j, d, a, c, i\}$.
   e) Draw the subgraph of **G** induced by $\{c, h, f, i, j\}$.
   f) Draw a subgraph of **G** having vertex set $\{e, f, b, c, h, j\}$ that is *not* an induced subgraph.
   g) Draw a spanning subgraph of **G** with exactly 10 edges.

6. Prove that every tree on n vertices has exactly $n - 1$ edges.

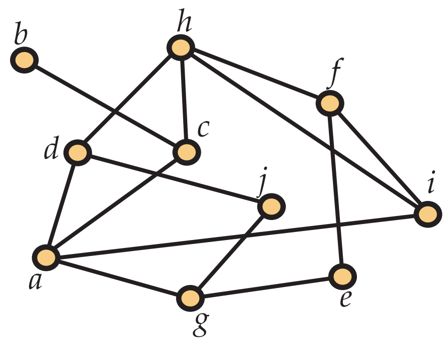


Figure 5.29.: A GRAPH **G**

7. Figure 5.30 contains four graphs on six vertices. Determine which (if any) pairs of graphs are isomorphic. For pairs that are isomorphic, give an isomorphism between the two graphs. For pairs that are not isomorphic, explain why.

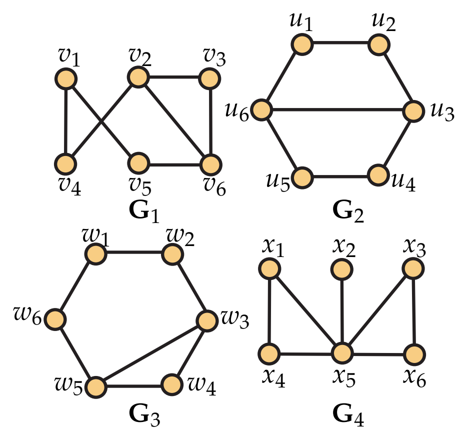


Figure 5.30.: ARE THESE GRAPHS ISOMORPHIC?

8. Find an eulerian circuit in the graph **G** in Figure 5.31 or explain why one does not exist.

9. Consider the graph **G** in Figure 5.32. Determine if the graph is eulerian. If it is, find an eulerian circuit. If it is not, explain why it is not. Determine if the graph is hamiltonian. If it is, find a hamiltonian cycle. If it is not, explain why it is not.

10. Explain why the graph **G** in Figure 5.33 does not have an eulerian circuit, but show that by adding a single edge, you can make it eulerian.

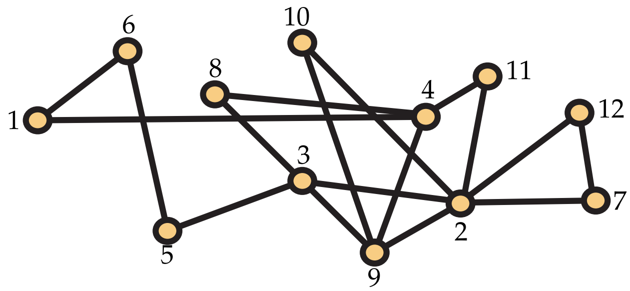


Figure 5.31.: A GRAPH **G**

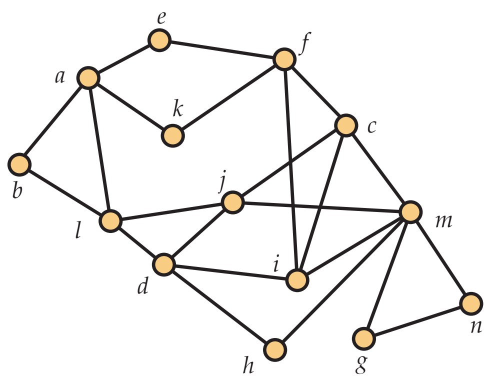


Figure 5.32.: A GRAPH **G**

11. An *eulerian trail* is defined in the same manner as an euler circuit (see page 67) except that we drop the condition that $x_0 = x_t$. Prove that a graph has an eulerian trail if and only if it is connected and has at most two vertices of odd degree.

12. Alice and Bob are discussing a graph that has 17 vertices and 129 edges. Bob argues that the graph is Hamiltonian, while Alice says that he's wrong. Without knowing anything more about the graph, must one of them be right? If so, who and why, and if not, why not?

13. Find the chromatic number of the graph **G** in Figure 5.34 and a coloring using $\chi(\mathbf{G})$ colors.

14. Find the chromatic number of the graph **G** in Figure 5.35 and a coloring using $\chi(\mathbf{G})$ colors.

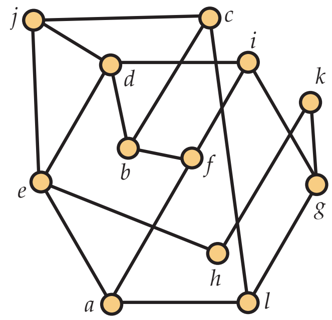


Figure 5.37.: Is this graph planar?

31. Draw a planar drawing of an eulerian planar graph with 10 vertices and 21 edges.

32. Show that every planar graph has a vertex that is incident to at most five edges.

33. Let $\mathbf{G} = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Its *degree sequence* is the list of the degrees of its vertices, arranged in nonincreasing order. That is, the degree sequence of $\mathbf{G}$ is $(\deg_{\mathbf{G}}(v_1), \deg_{\mathbf{G}}(v_2), \ldots, \deg_{\mathbf{G}}(v_n))$ with the vertices arranged such that $\deg_{\mathbf{G}}(v_1) \geq \deg_{\mathbf{G}}(v_2) \geq \cdots \geq \deg_{\mathbf{G}}(v_n)$. Below are five sequences of integers (along with n, the number of integers in the sequence). Identify

    - the *one* sequence that **cannot be the degree sequence of any graph**;
    - the *two* sequences that could be the degree sequence of a **planar** graph;
    - the *one* sequence that could be the degree sequence of a **tree**; i.e. with an Eulerian circuit
    - the *one* sequence that is the degree sequence of an **eulerian** graph; and
    - the *one* sequence that is the degree sequence of a graph that must be **hamiltonian**.

    Explain your answers. (Note that one sequence will get two labels from above.)

    a) $n = 10$: $(4, 4, 2, 2, 1, 1, 1, 1, 1, 1)$
    b) $n = 9$: $(8, 8, 8, 6, 4, 4, 4, 4, 4)$
    c) $n = 7$: $(5, 4, 4, 3, 2, 1, 0)$
    d) $n = 10$: $(7, 7, 6, 6, 6, 6, 5, 5, 5, 5)$
    e) $n = 6$: $(5, 4, 3, 2, 2, 2)$

34. Below are three sequences of length 10. One of the sequences cannot be the degree sequence (see exercise 33) of any graph. Identify it and say why. For each of the other two, say *why* (if you have enough information) a *connected* graph with that degree sequence

forms of some power series representations and specific coefficients for many more. However, unless an exercise specifically suggests that you use a computer algebra system, we strongly encourage you to solve the problem by hand. This will help you develop a better understanding of how generating functions can be used.

For all exercises in this section, "generating function" should be taken to mean "ordinary generating function." Exponential generating functions are only required in exercises specifically mentioning them.

1. For each *finite* sequence below, give its generating function.

   a) $1, 4, 6, 4, 1$
   b) $1, 1, 1, 1, 1, 0, 0, 1$
   c) $0, 0, 0, 1, 2, 3, 4, 5$
   d) $1, 1, 1, 1, 1, 1, 1$
   e) $3, 0, 0, 1, -4, 7$
   f) $0, 0, 0, 0, 1, 2, -3, 0, 1$

2. For each *infinite* sequence suggested below, give its generating function in closed form, i.e., *not* as an infinite sum. (Use the most obvious choice of form for the general term of each sequence.)

   a) $0, 1, 1, 1, 1, 1, \dots$
   b) $1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \dots$
   c) $1, 2, 4, 8, 16, 32, \dots$
   d) $0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \dots$
   e) $1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \dots$
   f) $2^8, 2^7\binom{8}{1}, 2^6\binom{8}{2}, \dots, \binom{8}{8}, 0, 0, 0, \dots$
   g) $1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, \dots$
   h) $0, 0, 0, 1, 2, 3, 4, 5, 6, \dots$
   i) $3, 2, 4, 1, 1, 1, 1, 1, 1, \dots$
   j) $0, 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, \dots$
   k) $6, 0, -6, 0, 6, 0, -6, 0, 6, \dots$
   l) $1, 3, 6, 10, 15, \dots, \binom{n+2}{2}, \dots$

3. For each generating function below, give a closed form for the n^{th} term of its associated sequence.

   a) $(1+x)^{10}$
   b) $\dfrac{1}{1-x^4}$
   c) $\dfrac{x^3}{1-x^4}$
   d) $\dfrac{1-x^4}{1-x}$
   e) $\dfrac{1+x^2-x^4}{1-x}$
   f) $\dfrac{1}{1-4x}$
   g) $\dfrac{1}{1+4x}$
   h) $\dfrac{x^5}{(1-x)^4}$
   i) $\dfrac{x^2+x+1}{1-x^7}$
   j) $3x^4 + 7x^3 - x^2 + 10 + \dfrac{1}{1-x^3}$

4. Find the coefficient on x^{10} in each of the generating functions below.
   a) $(x^3 + x^5 + x^6)(x^4 + x^5 + x^7)(1 + x^5 + x^{10} + x^{15} + \cdots)$
   b) $(1 + x^3)(x^3 + x^4 + x^5 + \cdots)(x^4 + x^5 + x^6 + x^7 + x^8 + \cdots)$

c) $(1+x)^{12}$

d) $\dfrac{x^5}{1-3x^5}$

e) $\dfrac{1}{(1-x)^3}$

f) $\dfrac{1}{1-5x^4}$

g) $\dfrac{x}{1-2x^3}$

h) $\dfrac{1-x^{14}}{1-x}$

5. Find the generating function for the number of ways to create a bunch of n balloons selected from white, gold, and blue balloons so that the bunch contains at least one white balloon, at least one gold balloon, and at most two blue balloons. How many ways are there to create a bunch of 10 balloons subject to these requirements?

6. A volunteer coordinator has 30 identical chocolate chip cookies to distribute to six volunteers. Use a generating function (and computer algebra system) to determine the number of ways she can distribute the cookies so that each volunteer receives at least two cookies and no more than seven cookies.

7. Consider the inequality
$$x_1 + x_2 + x_3 + x_4 \leq n$$
where $x_1, x_2, x_3, x_4, n \geq 0$ are all integers. Suppose also that $x_2 \geq 2$, x_3 is a multiple of 4, and $1 \leq x_4 \leq 3$. Let c_n be the number of solutions of the inequality subject to these restrictions. Find the generating function for the sequence $\{c_n : n \geq 0\}$ and use it to find a closed formula for c_n.

8. Find the generating function for the number of ways to distribute blank scratch paper to Alice, Bob, Carlos, and Dave so that Alice gets at least two sheets, Bob gets at most three sheets, the number of sheets Carlos receives is a multiple of three, and Dave gets at least one sheet but no more than six sheets of scratch paper. Without finding the power series expansion for this generating function (or using a computer algebra system!), determine the coefficients on x^2 and x^3 in this generating function.

9. What is the generating function for the number of ways to select a group of n students from a class of p students? $\rightarrow$ (for fixed p)

10. Using generating functions, find a formula for the number of different types of fruit baskets containing of n pieces of fruit chosen from pomegranates, bananas, apples, oranges, pears, and figs that can be made subject to the following restrictions:

   - there are either 0 or 2 pomegranates,
   - there is at least 1 banana,
   - the number of figs is a multiple of 5,
   - there are at most 4 pears, and

18. What is the smallest integer that can be partitioned in at least 1000 ways? How many ways can it be partitioned? How many of them are into distinct parts? (A computer algebra system will be helpful for this exercise.)

19. What is the generating function for the number of partitions of an integer into even parts?

20. Find the exponential generating function (in closed form, not as an infinite sum) for each infinite sequence $\{a_n : n \geq 0\}$ whose general term is given below.

    a) $a_n = 5^n$
    b) $a_n = (-1)^n 2^n$
    c) $a_n = 3^{n+2}$
    d) $a_n = n!$
    e) $a_n = n$
    f) $a_n = 1/(n+1)$

21. For each exponential generating function below, give a formula in closed form for the sequence $\{a_n : n \geq 0\}$ it represents.

    a) e^{7x}
    b) $x^2 e^{3x}$
    c) $\dfrac{1}{1+x}$
    d) e^{x^4}

22. Find the coefficient on $x^{10}/10!$ in each of the exponential generating functions below.

    a) e^{3x}
    b) $\dfrac{e^x - e^{-x}}{2}$
    c) $\dfrac{e^x + e^{-x}}{2}$
    d) $xe^{3x} - x^2$
    e) $\dfrac{1}{1-2x}$
    f) e^{x^2}

23. Find the exponential generating function for the number of strings of length n formed from the set $\{a, b, c, d\}$ if there must be at least one a and the number of c's must be even. Find a closed formula for the coefficients of this exponential generating function.

24. Find the exponential generating function for the number of strings of length n formed from the set $\{a, b, c, d\}$ if there must be at least one a and the number of c's must be odd. Find a closed formula for the coefficients of this exponential generating function.

25. Find the exponential generating function for the number of strings of length n formed from the set $\{a, b, c, d\}$ if there must be at least one a, the number of b's must be odd, and the number of d's is either 1 or 2. Find a closed formula for the coefficients of this exponential generating function.

Theorem 9.25. *The generating function for the number c_n of rooted, unlabeled, binary, ordered trees with n leaves is*

$$C(x) = \frac{1-\sqrt{1-4x}}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n.$$

Notice that c_n is a Catalan number, which we first encountered in chapter 2, where we were counting lattice paths that did not cross the diagonal line $y = x$. (The coefficient c_n is the Catalan number we called $C(n-1)$ in chapter 2.)

9.8. Discussion

Yolanda took a sip of coffee "I'm glad I paid attention when we were studying vector spaces, bases and dimension. All this stuff about solutions for recurrence equations made complete sense. And I can really understand why the professor was making a big deal out of factoring. We saw it our first semester when we were learning about partial fractions in calculus. And we saw it again with the differential equations stuff. Isn't it really neat to see how it all fits together." All this enthusiasm was too much for Alice who was not having a good day. Bob was more sympathetic "Except for the detail about zero as a root of an advancement operator polynomial, I was ok with this chapter." Xing said "Here we learned a precise approach that depended only on factoring. I've been reading on the web and I see that there have been some recent breakthroughs on factoring." Bob jumped back in "But even if you can factor like crazy, if you have a large degree polynomial in the advancement operator equation, then you will have lots of initial conditions. This might be a second major hurdle." Dave mumbled "Just do the factoring. The rest is easy." Carlos again was quiet but he knew that Dave was right. Solving big systems of linear equations is relatively easy. The challenge is in the factoring stage.

9.9. Exercises

1. Write each of the following recurrence equations as advancement operator equations.

   a) $r_{n+2} = r_{n+1} + 2r_n$
   b) $r_{n+4} = 3r_{n+3} - r_{n+2} + 2r_n$
   c) $g_{n+3} = 5g_{n+1} - g_n + 3^n$
   d) $h_n = h_{n-1} - 2h_{n-2} + h_{n-3}$
   e) $r_n = 4r_{n-1} + r_{n-3} - 3r_{n-5} + (-1)^n$
   f) $b_n = b_{n-1} + 3b_{n-2} + 2^{n+1} - n^2$

2. Solve the recurrence equation $r_{n+2} = r_{n+1} + 2r_n$ if $r_0 = 1$ and $r_2 = 3$ (Yes, we specify a value for r_2 but not for r_1).

3. Find the general solution of the recurrence equation $g_{n+2} = 3g_{n+1} - 2g_n$.
4. Solve the recurrence equation $h_{n+3} = 6h_{n+2} - 11h_{n+1} + 6h_n$ if $h_0 = 3$, $h_1 = 2$, and $h_2 = 4$.
5. Find an explicit formula for the n^{th} Fibonacci number f_n. (See subsection 9.1.1.)
6. For each advancement operator equation below, give its general solution.

   a) $(A-2)(A+10)f = 0$
   b) $(A^2-36)f = 0$
   c) $(A^2-2A-5)f = 0$
   d) $(A^3-4A^2-20A+48)f = 0$
   e) $(A^3+A^2-5A+3)f = 0$
   f) $(A^3+3A^2+3A+1)f = 0$

7. Solve the advancement operator equation $(A^2+3A-10)f = 0$ if $f(0) = 2$ and $f(1) = 10$.
8. Give the general solution to each advancement operator equation below.
   a) $(A-4)^3(A+1)(A-7)^4(A-1)^2 f = 0$
   b) $(A+2)^4(A-3)^2(A-4)(A+7)(A-5)^3 g = 0$
   c) $(A-5)^2(A+3)^3(A-1)^3(A^2-1)(A-4)^3 h = 0$
9. For each nonhomogeneous advancement operator equation, find its general solution.

   a) $(A-5)(A+2)f = 3^n$
   b) $(A^2+3A-1)g = 2^n + (-1)^n$
   c) $(A-3)^3 f = 3n+1$
   d) $(A^2+3A-1)g = 2n$
   e) $(A-2)(A-4)f = 3n^2 + 9^n$
   f) $(A+2)(A-5)(A-1)f = 5^n$
   g) $(A-3)^2(A+1)g = 2 \cdot 3^n$
   h) $(A-2)(A+3)f = 5n2^n$
   i) $(A-2)^2(A-1)g = 3n^2 2^n + 2^n$
   j) $(A+1)^2(A-3)f = 3^n + 2n^2$

i.e. consisting of 0,1,2

10. Find ~~and solve~~ a recurrence equation for the number g_n of ternary strings of length n that do not contain 102 as a substring.
11. There is a famous puzzle called the Towers of Hanoi that consists of three pegs and n circular discs, all of different sizes. The discs start on the leftmost peg, with the largest disc on the bottom, the second largest on top of it, and so on, up to the smallest disc on top. The goal is to move the discs so that they are stacked in this same order on the rightmost peg. However, you are allowed to move only one disc at a time, and you are never able to place a larger disc on top of a smaller disc. Let t_n denote the fewest moves (a move being taking a disc from one peg and placing it onto another) in which you can accomplish the goal. Determine an explicit formula for t_n.

12. A valid database identifier of length n can be constructed in three ways:
    - Starting with A and followed by any valid identifier of length $n-1$.
    - Starting with one of the two-character strings $1A$, $1B$, $1C$, $1D$, $1E$, or $1F$ and followed by any valid identifier of length $n-2$.
    - Starting with 0 and followed by any ternary ($\{0,1,2\}$) string of length $n-1$.

    Find a recurrence for the number $g(n)$ of database identifiers of length n and then solve your recurrence to obtain an explicit formula for $g(n)$. (You may consider the empty string of length 0 a valid database identifier, making $g(0)=1$. This will simplify the arithmetic.)

13. Let t_n be the number of ways to tile a $2 \times n$ rectangle using 1×1 tiles and L-tiles. An L-tile is a 2×2 tile with the upper-right 1×1 square deleted. (An L tile may be rotated so that the "missing" square appears in any of the four positions.) Find a recursive formula for t_n along with enough initial conditions to get the recursion started. Use this recursive formula to find a closed formula for t_n.

14. Prove Lemma 9.20 about advancement operator equations with repeated roots.

15. Use generating functions to solve the recurrence equation $r_n = 4r_{n-1} + 6r_{n-2}$ for $n \geq 2$ with $r_0 = 1$ and $r_1 = 3$.

16. Let $a_0 = 0$, $a_1 = 2$, and $a_2 = 5$. Use generating functions to solve the recurrence equation $a_{n+3} = 5a_{n+2} - 7a_{n+1} + 3a_n + 2^n$ for $n \geq 0$.

17. Let $b_0 = 1$, $b_2 = 1$, and $b_3 = 4$. Use generating functions to solve the recurrence equation $b_{n+3} = 4b_{n+2} - b_{n+1} - 6b_n + 3^n$ for $n \geq 0$.

18. Use generating functions to find a closed formula for the Fibonacci numbers f_n.

19. How many rooted, unlabeled, binary, ordered, trees (RUBOTs) with 6 leaves are there? Draw 6 distinct RUBOTs with 6 leaves.

20. In this chapter, we developed a generating function for the Catalan numbers. We first encountered the Catalan numbers in chapter 2, where we learned they count certain lattice paths. Develop a recurrence for the number l_n of lattice paths similar to the recurrence

$$c_n = \sum_{k=0}^{n} c_k c_{n-k} \qquad \text{for } n \geq 2$$

    for RUBOTs by thinking of ways to break up a lattice path from $(0,0)$ to (n,n) that does not cross the diagonal $y = x$ into two smaller lattice paths of this type.

4. A linear programming problem posed with integer coefficients and constants need not have an optimal solution with integer values—even when it has an optimal solution with rational values.

5. A very important theme in operations research is to determine when a linear programming problem posed in integers has an optimal solution with integer values. This is a subtle and often very difficult problem.

6. The problem of finding a maximum flow in a network is a special case of a linear programming problem.

7. A network flow problem in which all capacities are integers has a maximum flow in which the flow on every edge is an integer. The Ford-Fulkerson labeling algorithm guarantees this!

8. In general, linear programming algorithms are not used on networks. Instead, special purpose algorithms, such as Ford-Fulkerson, have proven to be more efficient in practice.

13.7. Exercises

1. Consider the network diagram in Figure 13.6. For each directed edge, the first number is the capacity and the second value is intended to give a flow ϕ in the network. However, the flow suggested is not valid.
   a) Identify the reason(s) ϕ is not valid.
   b) Without changing any of the edge capacities, modify ϕ into a valid flow $\widehat{\phi}$. Try to use as few modifications as possible.

2. Alice claims to have found a (valid) network flow of value 20 in the network shown in Figure 13.7. Bob tells her that there's no way she's right, since no flow has value greater than 18. Who's right and why? → Consider the cut (P, P') where P = all vertices except sink and $P' = \{\text{sink}\}$

3. Find an augmenting path P with *at least one backward edge* for the flow ϕ in the network shown in Figure 13.8. What is the value of δ for P? Carry out an update of ϕ using P to obtain a new flow $\hat{\phi}$. What is the value of $\hat{\phi}$?

4. Prove Proposition 13.6. You will need to verify that the flow conservation laws hold at each vertex along an augmenting path (other than S and T). There are four cases to consider depending on the forward/backward status of the two edges on the augmenting path that are incident with the vertex.

5. Find the capacity of the cut (L, U) with

$$L = \{S, F, H, C, B, G, I\} \qquad \text{and} \qquad U = \{A, D, E, T\}$$

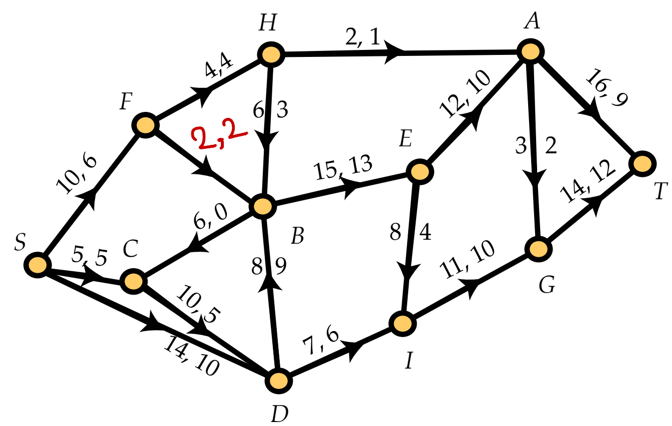


Figure 13.6.: An invalid flow in a network

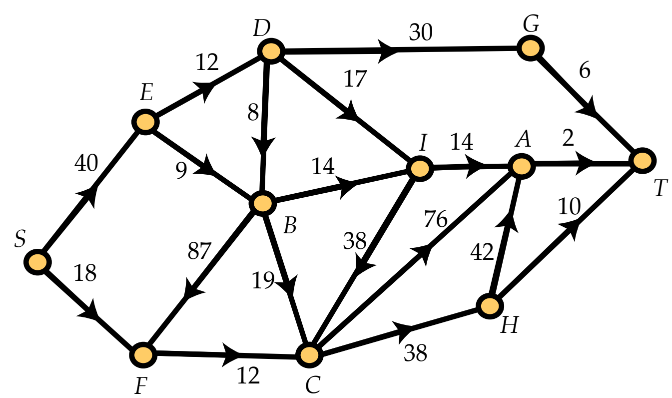


Figure 13.7.: A network

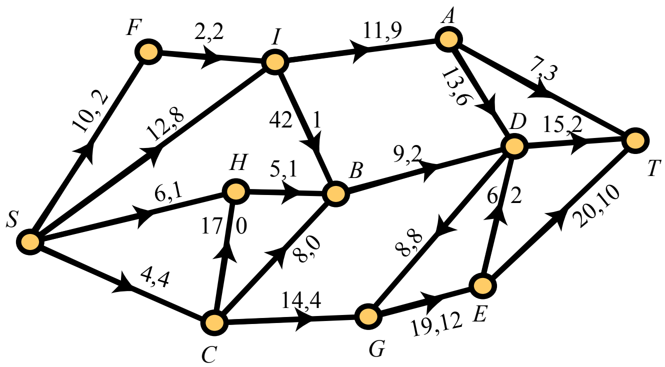


Figure 13.8.: A NETWORK WITH FLOW

in the network shown in Figure 13.8.

6. Find the capacity of the cut (L, U) with

$$L = \{S, F, D, B, A\} \qquad \text{and} \qquad U = \{H, C, I, G, E, T\}$$

in the network shown in Figure 13.8.

7. For each of the augmenting paths P_1, P_2, P_3, and P_4 in Example 13.7, update the flow in Figure 13.2. (Note that your solution to this exercise should consist of four network flows. Do not attempt to use the four paths in sequence to create one updated network flow.)

8. Continue running the Ford-Fulkerson labeling algorithm on the network flow in Figure 13.4 until the algorithm halts without labeling the sink. Find the value of the maximum flow as well as a cut of minimum capacity.

9. Use the Ford-Fulkerson labeling algorithm to find a maximum flow and a minimum cut in the network shown in Figure 13.9 by starting from the current flow shown there.

10. Figure 13.10 shows a network. Starting from the zero flow, i.e., the flow with $\phi(e) = 0$ for every directed edge e in the network, use the Ford-Fulkerson labeling algorithm to find a maximum flow and a minimum cut in this network.

11. Consider a network in which the source S has precisely three neighbors: B, E, and F. Suppose also that $c(S, B) = 30$, $c(S, E) = 20$, and $c(S, F) = 25$. You know that there is a flow ϕ on the network but you do not know how much flow is on any edge. You do know, however, that when the Ford-Fulkerson labeling

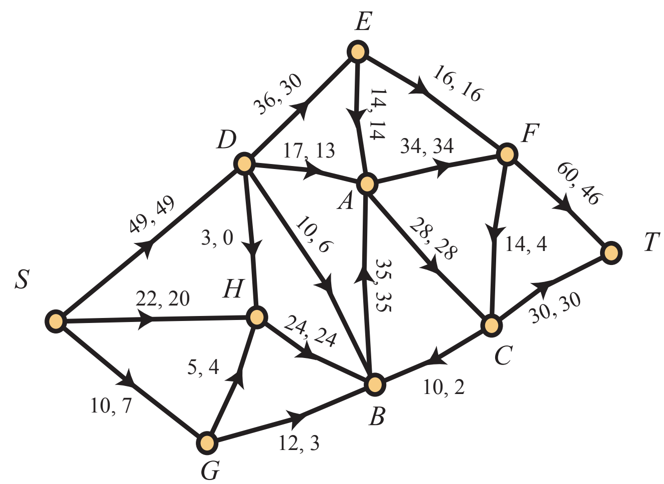


Figure 13.9.: A network with flow

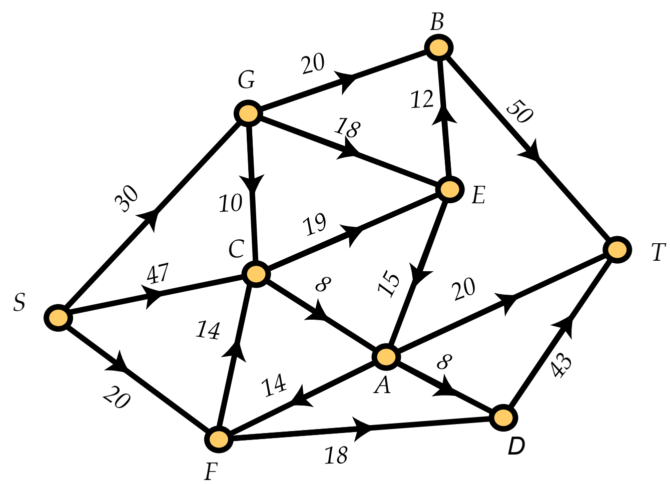


Figure 13.10.: A network

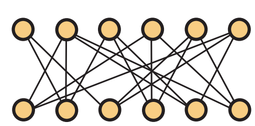
to find an antichain of as many points as there are chains in our partition. (In the example we've been using, we need to find a three-element antichain.) This is where tracking the labeled vertices comes in handy. Suppose we have determined a chain $C = \{x_1 < x_2 < \cdots < x_k\}$ using the network flow. Since x_1 is the minimal element of this chain, there is no flow into x_1'' and hence no flow out of x_1''. Since T is unlabeled, this must mean that x_1'' is unlabeled. Similarly, x_k is the maximal element of C, so there is no flow out of x_k'. Thus, x_k' is labeled. Now considering the sequence of vertices

$$x_k', x_k'', x_{k-1}', x_{k-1}'', \ldots, x_2', x_2'', x_1', x_1'',$$

there must be a place where the vertices switch from being labeled to unlabeled. This must happen with x_i' labeled and x_i'' unlabeled. To see why, suppose that x_i' and x_i'' are both unlabeled while x_{i+1}' and x_{i+1}'' are both labeled. Because x_i and x_{i+1} are consecutive in C, there is flow on (x_i', x_{i+1}''). Therefore, when scanning from x_{i+1}'', the vertex x_i' would be labeled. For each chain of the chain partition, we then take the first element y for which y' is labeled and y'' is unlabeled to form an antichain $A = \{y_1, \ldots, y_w\}$. To see that A is an antichain, notice that if $y_i < y_j$, then (y_i', y_j'') is an edge in the network. Therefore, the scan from y_i' would label y_j''. Using this process, we find that a maximum antichain in our example is $\{x_1, x_5, x_8\}$.

14.4. Exercises

1. Use the techniques of this chapter to find a ~~maximum~~ Complete (perfect) matching from V_1 to V_2 in the graph shown in Figure 14.9. The vertices on the bottom are the set V_1, while the vertices on the top are the set V_2. If you cannot find a matching that saturates all of the vertices in V_1, explain why. → Phillip Hall's theorem



V_2

V_1

Figure 14.9.: Is there a matching saturating V_1?

2. Use the techniques of this chapter to find a ~~maximum~~ Complete (perfect) matching from V_1 to V_2 in the graph shown in Figure 14.10. The vertices on the bottom are the set V_1, while the vertices on the top are the set V_2. If you cannot find a matching that saturates all of the vertices in V_1, explain why.

3. Students are preparing to do final projects for an applied combinatorics course. The five possible topics for their final projects are graph algorithms, posets, induction, graph theory, and generating functions. There are five students in the

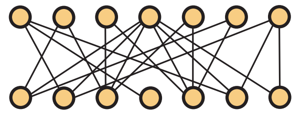


Figure 14.10.: Is there a matching saturating V_1?

class, and they have each given their professor the list of topics on which they are willing to do their project. Alice is interested in posets or graphs. Bob would be willing to do his project on graph algorithms, posets, or induction. Carlos will only consider posets or graphs. Dave likes generating functions and induction. Yolanda wants to do her project on either graphs or posets. To prevent unauthorized collaboration, the professor does not want to have two students work on the same topic. Is it possible to assign each student a topic from the lists above so that no two students work on the same project? If so, find such an assignment. If not, find an assignment that maximizes the number of students who have assignments from their lists and explain why you cannot satisfy all the students' requests.

4. Seven colleges and universities are competing to recruit six high school football players to play for their varsity teams. Each school is only allowed to sign one more player, and each player is only allowed to commit to a single school. The table below lists the seven institutions and the students they are trying to recruit, have been admitted, and are also interested in playing for that school. (There's no point in assigning a school a player who cannot meet academic requirements or doesn't want to be part of that team.) The players are identified by the integers 1 through 6. Find a way of assigning the players to the schools that maximizes the number of schools who sign one of the six players.

School	Player numbers
Boston College	1, 3, 4
Clemson University	1, 3, 4, 6
Georgia Institute of Technology	2, 6
University of Georgia	None interested
University of Maryland	2, 3, 5
University of North Carolina	1, 2, 5
Virginia Polytechnic Institute and State University	1, 2, 5, 6

5. The questions in this exercise refer to the network diagram in Figure 14.11. This network corresponds to a poset **P**. As usual, all capacities are assumed to be 1, and all edges are directed upward. Answer the following questions about **P** *without drawing the diagram of the poset*.

