# MATH 2370, Practice Problems

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**Problem:** Prove that an $n \times n$ complex matrix $A$ is diagonalizable if and only if there is a basis consisting of eigenvectors of $A$.

**Problem:** Let $A : V \to W$ be a one-to-one linear map between two finite dimensional vector spaces $V$ and $W$. Show that the dual map $A' : W' \to V'$ is surjective.

**Problem:** Determine if the curve

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + xy = 10\}$$

is an ellipse or hyperbola or union of two lines.

**Problem:** Show that if a nilpotent matrix is diagonalizable then it is the zero matrix.

**Problem:** Let $P$ be a permutation matrix. Show that $P$ is diagonalizable. Show that if $\lambda$ is an eigenvalue of $P$ then for some integer $m > 0$ we have $\lambda^m = 1$ (i.e. $\lambda$ is an $m$-th root of unity). Hint: Note that $P^m = I$ for some integer $m > 0$.

**Problem:** Show that if $\lambda$ is an eigenvector of an orthogonal matrix $A$ then $|\lambda| = 1$.

**Problem:** Take a vector $v \in \mathbb{R}^n$ and let $H$ be the hyperplane orthogonal to $v$. Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection with respect to a hyperplane $H$. Prove that $R$ is a diagonalizable linear map.

**Problem:** Prove that if $\lambda_1, \lambda_2$ are distinct eigenvalues of a complex matrix $A$ then the intersection of the generalized eigenspaces $E_{\lambda_1}$ and $E_{\lambda_2}$ is zero (this is part of the Spectral Theorem).

**Problem:** Let $H = (h_{ij})$ be a $2 \times 2$ Hermitian matrix. Use the Minimax Principle to show that if $\lambda_1 \leq \lambda_2$ are the eigenvalues of $H$ then $\lambda_1 \leq h_{11} \leq \lambda_2$.

**Problem:** Suppose $M$ is a real symmetric $n \times n$ matrix. Prove that if all the eigenvalues of $M$ are positive then for any $0 \neq x \in \mathbb{R}^n$ we have $(x, Mx) > 0$. Here $(\cdot, \cdot)$ is the standard scalar product on $\mathbb{R}^n$. (As you may know such an $M$ is called *positive definite*.)

**Problem:** Let $A, B$ be $n \times n$ complex matrices. Recall that the Hilbert-Schmidt norm $||A||_{HS}$ of $A$ is $tr(A^*A)$. Prove that:

$$||AB||_{HS} \leq ||A||_{HS}||B||_{HS}.$$

(Hint: By the Cauchy-Schwarz inequality we know $||Ax|| \leq ||A||_{HS}||x||$ for all $x$. Apply this and take $x$ to be columns of $B$.)

**Problem:** Let $A$ be an $n \times n$ complex matrix. Prove that $det(A)$ is product of all the eigenvalues of $A$ (where each eigenvalue is repeated as many times as it appears in the characteristic polynomial). Similarly, show that $tr(A)$ is the sum of eigenvalues of $A$.

**Problem:** Let $A$ be a normal matrix. Show that $||A|| = r(A)$, where $||A||$ is the operator norm of $A$ and $r(A)$ is the spectral radius.

**Problem:** Prove that $r(A) = r(A^T)$.

**Problem:** Compute the operator norm of the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

**Problem:** Give an example of a diagonalizable matrix $A$ such that $||A|| \neq r(A)$.

**Problem:** What is the operator norm, Hilbert-Schmidt norm and spectral radius of a unitary matrix?

**Problem:** Let $A$ be an $n \times n$ complex matrix. Show that all the eigenvalues of $AA^*$ are non-negative real numbers. Here $A$ is the adjoint of $A$

with respect to the standard complex scalar product (Hermitian product), i.e. $A^* = \bar{A}^T$.

**Problem:** Give an example of a real symmetric $2 \times 2$ matrix $M$ and an invertible $2 \times 2$ matrix $L$ such that $D = L^T M L$ is diagonal but the diagonal entries of $D$ are not eigenvalues of $M$.

**Problem:** Let $M$ be a Hermitian matrix with non-negative eigenvalues. Show that $M = A^* A$ for some $n \times n$ complex matrix $A$. (Hint: first assume $M$ is diagonal.)

**Problem:** Suppose $A$ is a Hermitian matrix such that all its eigenvalues are non-negative. Show that $A$ has a square root, that is, there is an $n \times n$ matrix $B$ with $B^2 = A$. Moreover, show that $B$ can be taken to be Hermitian and with non-negative eigenvalues.

**Problem:** Use Gram-Schmidt orthonormalization to show that any invertible real matrix $A$ can be written as $A = KB$ where $K$ is an orthogonal matrix and $B$ is upper triangular (similarly any complex invertible matrix is the product of a unitary and an upper triangular).

**Problem:** Prove that for any $n \times n$ complex matrix $A$ we have

$$tr(A^* A) \leq n||A||^2.$$

(Hint: recall that $tr(A^*A) = \sum_{ij} |a_{ij}|^2$. Use the definition of $||A||$ and consider $||Ae_i||$ for all standard basis elements $e_i$.)

**Problem:** Let $\Pi$ be the plane in $\mathbb{R}^3$ defined by $x + 2y + z = 0$. Let $T$ be the reflection in $\mathbb{R}^3$ with respect to the plane $\Pi$. Find the matrix representation of $T$ (with respect to the standard basis).

**Problem:** Let $P$ be an $n \times n$ complex matrix such that $P^2 = P$. and $rank(P) = r$. Show that $P$ is similar to:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

**Problem:** Let $A$, $B : \mathbb{C}^n \to \mathbb{C}^n$ be two orthogonal projections satisfying:

$$||Ax||^2 + ||Bx||^2 = ||x||^2,$$

for any $x \in \mathbb{C}^n$. Prove that $A + B = I$. (Hint: let $A$ and $B$ be orthogonal projections on the subspaces $U$ and $W$ respectively. It suffices to show that $U \oplus W = \mathbb{C}^n$ and $U$ is orthogonal to $W$.)

**Problem:** Let $A$ be an $n \times n$ matrix. Show that the null space of $A$ is the same as the null space of $A^T A$.

**Problem:** Let $A$, $B$ be $n \times n$ matrices. Show that:

$$det(I - AB) = det(I - BA).$$

**Problem:** Let $A$ be a $4 \times 4$ symmetric matrix. Suppose $A^2 + 3A = 0$ and that $rank(A) = 2$. Find the characteristic polynomial of $A$. Find $tr(A)$ and $det(A)$.

**Problem:** Prove that a unitary matrix is diagonalizable.

**Problem:** Suppose $A$ and $B$ are normal matrices. Show that:

$$r(AB) \leq r(A)r(B).$$