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Problem 1.

Suppose W_1 and W_2 are complements in a complex inner product space V. Prove that $W_1^{\perp}$ and $W_2^{\perp}$ are also complements.

Problem 2.

a) Suppose A is a diagonalizable complex matrix. Prove that $A^2 - A$ is also diagonalizable.

b) Is the opposite true? Prove or provide a counterexample.

Problem 3.

Suppose f is a polynominal with complex coefficients, A and B are complex matrices.

a) If $A = f(B)$ then A an B commute.

b) If A and B are commuting 2x2 matrices, then for some f either $A = f(B)$ or $B = f(A)$.

c) This is false for 3x3 matrices.

Problem 4.

Suppose W is a subspace of a real Euclidean space V.

a) Suppose w is a vector in W, v is a vector in V. Then the angle between v and W is acute iff the angle between $proj_W(v)$ and w is.

b) Suppose u and v are two vectors in V, and and u_1 and v_1 are their projections to W. Prove that neither of the statements (angle between u and v is acute) and (angle between u_1 and v_1 is acute) implies the other.

c) Suppose v and w are as in a). Prove that the angle between $proj_W(v)$ and w is less than or equal to that between w and v.

Problem 5.

Suppose $v_1, v_2, ...$ are eigenvectors of some operator A.

a) Suppose $v_1 + v_2 = v_3$. Prove that the all have the same eigenvalue.

b) Suppose $v_1 + ... + v_n = v_{n+1}$. Prove that some of $v_1, ...v_n$ have the same eigenvalue.

c) Suppose $v_1 + ... + v_n = v_{n+1}$. Prove that either all of these vectors have the same eigenvalue or there is a subset of vectors $\{v_1, ..., v_n\}$ that adds up to zero.

Problem 6.

Suppose V is a finite-dimensional complex inner product space. Suppose W and $W^{\perp}$ are subspaces of V, invariant under an operator A.

Denote by A_1 and A_2 the restrictions of A to W and $W^\perp$ respectively. Prove that

$$||A|| = \max\{||A_1||, ||A_2||\}$$

Problem 7.

Suppose A and B are positive-definite real matrices. Suppose

$$||A + B|| = ||A|| + ||B||$$

Show that A and B have a common eigenvector.

Problem 8.

Suppose V_1 and V_2 are subspaces of a finite-dimensional vector space V. Prove that

$$\dim(V_1 + V_2) + \dim(V_1 \cap V_2) = \dim V_1 + \dim V_2.$$

Problem 9.

Suppose A and B are commuting nilpotent matrices.

a) Prove that $A + B$ is nilpotent

b) Prove that AB is nilpotent

c) Give a counterexample to a) and b) if the nilpotent matrices do not commute.

Problem 10.

Suppose A has one Jordan block, and B commutes with A. Prove that there exists a polynomial f with complex coefficients so that $B = f(A)$.

Problem 11.

Suppose A and B are commuting operators on a finite-dimensional space V. We know that $A = A_d + A_n$, where A_d is diagonalizable, A_n is nilpotent and $A_dA_n = A_nA_d$. Likewise, $B = B_d + B_n$. Prove that A_d, A_n, B_d, B_n commute with each other.

Problem 12.

Suppose $AB = BA = A^2$, and A is diagonalizable. Prove that if B is a projection, then A is a projection too.

Problem 13.

Suppose A and B are 2×2 matrices. Suppose that

$$AB + BA = 0$$

Prove that $AB = 0$ or $\mathrm{tr}A = \mathrm{tr}B = 0$.

Problem 14.

Suppose A has rank one. Prove that A is either diagonalizable or nilpotent.

Problem 15.

Suppose $A_1, ..., A_{n+1}$ are pairwise commuting $n \times n$ matrices. Suppose

$$A_1 A_2 ... A_{n+1} = 0$$

Prove that in the above expression at least one of the factors can be removed with the expression still being zero.

Problem 16.

Prove that a complex matrix A is normal if and only if $A^* = f(A)$ for some polynomial f.

Problem 17.

a) Suppose $W \subset V$ is a codimension one subspace. Suppose u and v are vectors in V so that they are orthogonal and their projections to W are orthogonal. Prove that at least one of these vectors belongs to W.

b) Suppose $W \subset V$ is a subspace of codimension more than one. Prove that one can find non-zero vectors u and v in V that are orthogonal and $(proj_W u, proj_W v) = 0$. Additionally, if the dimension of W is at least two, one can require the projections to be non-zero.

Problem 18.

Suppose $L : V \to V$, V is an inner product space (Euclidean or Hermitian). Suppose $W \subset V$ is a subspace, invariant under L.

Prove that L is self-adjoint if and only if the following three conditions are all satisfied:

1) $W^\perp$ is invariant under L;

2) the restriction of L to W is self-adjoint;

3) the restriction of L to $W^\perp$ is self-adjoint.

Problem 19. (Universal Property of Quitients)

Suppose W is a subspace of V, and H is another linear vector space. Suppose $A : V \to H$ is a linear map such that $L(W) = \{0\}$. Prove that there exists a linear map $B : V/W \to H$ so that $A = B \circ \pi$, where

$\pi : V \to V/W$ is the natural quotient map. Prove that this map B is unique.

Problem 20.

Suppose A is an $n \times n$ matrix, which is real, anti-self-adjoint and invertible. Prove that n is even.

Problem 21.

Suppose A is an $n \times n$ real matrix, with no real invariant one-dimensional subspaces. Prove that n is even.

Find an example of a 2×2 real matrix with no real invariant one-dimensional subspaces.

Problem 22.

Suppose V is the space of polynomials in the variable x of degree at most three with the inner product defined as follows:

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\; dx.$$

Construct an orthonormal basis of V.

Problem 23.

Suppose M is the space of 2×2 matrices with the inner produt defined as follows:

$$(A, B) = tr\ (AB^*).$$

a) Construct an orthonormal basis of M.

b) Construct an orthonormal basis of M, consisting of invertible matrices.

c) Prove that it is impossible to construct an orthonormal basis of M, consisting of nilpotent matrices.

Problem 24.

Suppose A, B, and $A + B$ are unitary.

Prove that $(AB^*)^3 = I$.

Problem 25.

Suppose A, B, and $A + B$ are nilpotent $n \times n$ matrices.

a) If $n = 2$, prove that $AB = 0$.

b) Find an example for $n = 3$ with $AB \neq 0$.

c) Find an example for $n = 3$ with $AB \neq BA$.

Problem 26.

Suppose $A+B=C$, where A and B are unitary and C is a nilpotent 2×2 matrix. Prove that $C=0$ or A^2 is a scalar matrix.

Problem 27.

Suppose A and B are 3×3 complex matrices, and $A^2=B^2=0$. Prove that A and B have a common eigenvector.

Problem 28.

Suppose $T:\mathbb{C}^n\to\mathbb{C}^n$ is a map which rotates the basis, i.e. for the basic vectors e_i, $i=1,...,n-1$, we have $Te_i=e_{i+1}$, and $Te_n=e_1$.

a) Prove that T is unitary.

b) Find all eigenvalues of T.

c) Describe all eigenvectors of T.

Problem 29.

Suppose A is a 3×3 real orthogonal matrix with $det\ A=-1$.

a) Prove that -1 is an enigenvalue of A.

b) Describe geometrically all such matrices A.

Problem 30.

Suppose A is an invertible complex matrix. Prove that $A^{-1}=f(A)$, where $f(x)$ is a polynomial. If A is real, then the coefficients of f can be chosen to be real.

Problem 31.

Suppose $A,\ B,\ C,\ D$ are 2×2 matrices. Prove that some non-trivial linear combination of A and B commutes with a non-trivial linear combination of C and D.

Problem 32.

Suppose $A:V\to V$ is a linear map, V is a linear real vector space. Prove that A is diagonalizable over real numbers if and only if it is self-adjoint with respect to some positive-definite inner product on V.

Problem 33.

Suppose $A:V\to W$ and $B:W\to V$ are two linear maps. Suppose $AB=I$. Prove that

a) $\dim W\le\dim V$;

b) $BA=I$ if and only if $\dim W=\dim V$.

Problem 34.

Find a singular value decomposition for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 35.

Suppose V_1 and V_2 are complements in a complex inner product space V. Prove that the following three conditions are equivalent.

1) V_1 and V_2 are orthogonal complements.

2) For every orthonormal basis of V_1 and for every orthonormal basis of V_2 putting them together produces an orthonormal basis of V.

3) For some orthonormal basis of V_1 and for some orthonormal basis of V_2 putting them together produces an orthonormal basis of V.

Problem 36.

Suppose A is a positive definite real symmetirc matrix, and B is a real symmetric matrix. Suppose $AB + BA$ is positive definite. Prove that B is positive definite.