Math 2500 Midterm Exam

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Question (1)**.**

(a) Give the definitions of the following: (i) An action of a group G on a set X. (ii) A solvable group. (iii) A composition series for a group G.

(b) State the Jordan-Holder theorem, and the 1st isomorphism theorem.

Solution. (a) (i) A group action of a group G on a set X is a map from $G \times X$ to X satisfying the following properties:

(1) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G, a \in A$, and

(2) $e \cdot a = a$, for all $a \in A$.

(ii) A group G is solvable if there is a chain of subgroups

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for $i = 0, 1, \cdots, s-1$.

(iii) In a group G a sequence of subgroups

$$\{e\} = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{k-1} \leq N_k = G$$

is called a composition series if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is a simple group, $0 \leq i \leq k-1$.

(b) (i) (Jordan-Holder) Let G be a finite group with $G \neq \{e\}$. Then

(1) G has a composition series and

(2) The composition factors in a composition series are unique, namely, if $\{e\} = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{r-1} \leq N_r = G$ and $\{e\} = M_0 \leq M_1 \leq M_2 \leq \cdots \leq M_{s-1} \leq M_s = G$ are two composition series of G, then $r = s$ and there is some permutation, π, of $\{1, 2, \cdots, r\}$ such that

$$M_{\pi(i)}/M_{\pi(i)-1} \cong N_i/N_{i-1}, \; 1 \leq i \leq r.$$

(ii) (1st isomorphism theorem) If $\varphi : G \to H$ is a homomorphism of groups, then $\ker \varphi \trianglelefteq G$ and

$$G/\ker \varphi \cong \varphi(G).$$

Question (2)**.** Prove the orbit-stabilizer theorem: let G be a group acting on a set X. Let $H = G_x$ be the stabilizer subgroup of $x \in X$. Then there is a one-to-one correspondence between the orbit $G \cdot x$ and the coset space G/H.

Solution. Let $\phi : G \cdot x \to G/G_x$ be a map from the orbit of x to the left coset of G_x defined as:

$$\forall g \in G : \phi(g \cdot x) = gG_x.$$

Suppose $g \cdot x = h \cdot x$ for some $g, h \in G$. Then

$$(h^{-1}g) \cdot x = h^{-1} \cdot (g \cdot x) = h^{-1} \cdot (h \cdot x) = (h^{-1}h) \cdot x = x \Longrightarrow h^{-1}g \in G_x. \tag{1}$$

So, $gG_x = hG_x$, which means ϕ is well-defined.
Let $\phi(g_1 \cdot x) = \phi(g_2 \cdot x)$ for some $g_1, g_2 \in G$.
Then

$$g_1G_x = g_2G_x \Longrightarrow g_2^{-1}g_1 \in G_x \Longrightarrow x = (g_2^{-1}g_1) \cdot x. \tag{2}$$

Thus

$$g_2 \cdot x = g_2 \cdot (g_2^{-1}g_1) \cdot x = g_1 \cdot x. \tag{3}$$

So ϕ is injective. By the definition, ϕ is surjective. Hence, it is a bijection.

Question (3)**.** Show that A_4 is not a simple group. Does A_4 have a subgroup of index 2?

Solution. (i)
Since $\{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4, we have A_4 is not a simple group.
(ii) No. If there exists $H \leq A_4$ such that $|H| = 6$. Since the index is 2, H is normal. Then for $x \in A_4 \setminus H$, xH has order 2, as an element of A_4/H. Hence $x^3H = x(xH)^2 = xH \neq H$. Particularly, $x^3 \neq e$. However, there are 8 elements in A_4 has order 3, which can't all in H since $|H| = 6 < 8$. This leads a contradiction.

Question (4)**.** Let X be a finite set and suppose G is an abelian subgroup of the symmetric group S_X that acts transitively on X. Show that for all $e \neq g \in G$ and all $x \in X$ we have $g \cdot x \neq x$. Deduce that $|G| = |X|$.

Solution. If there exists $g \in G$ such that for some $x \in X$, we have $g \cdot x = x$. Since the action is transitive then for each $y \in X$, there exists $h_y \in G$ such that $h_y \cdot y = x$. So,

$$\begin{aligned}(h_y^{-1}g) \cdot x = h_y^{-1} \cdot (g \cdot x) =& h_y^{-1} \cdot x = h_y^{-1} \cdot (h_y \cdot y) = y,\\ (gh_y^{-1}) \cdot x =& (gh_y^{-1}) \cdot (h_y \cdot y) = g \cdot y.\end{aligned} \tag{4}$$

Since G is abelian, we have $h_y^{-1}g = gh_y^{-1}$, which means for $\forall y \in X$, $g \cdot y = y$. So $g = e$.

For a fixed $x \in X$, define $\phi : G \to X$ by $\phi(g) = g \cdot x$. Then for each $y \in X$, since the action is transitive, there exists $g_1 \in G$ such that $\phi(g_1) = g_1 \cdot x = y$, which means ϕ is surjective. If $g_1 \cdot x = g_2 \cdot x$, then

$$(g_1^{-1} g_2) \cdot x = g_1^{-1} \cdot (g_2 \cdot x) = g_1^{-1} \cdot (g_1 \cdot x) = x \Longrightarrow g_1^{-1} g_2 = e \Longrightarrow g_1 = g_2. \tag{5}$$

So, ϕ is an injection. Hence ϕ is a bijection. We have $|G| = |X|$.

Question (5)**.** Find all the conjugacy classes in S_4.

Solution. Two permutations in S_4 are conjugate if and only if they have the same cycle type. So there are 5 conjugacy classes in S_4:

$$\begin{aligned}
(1):&\quad (1); & (6)\\
(2):&\quad (12), (13), (14), (23), (24), (34); & (7)\\
(3):&\quad (123), (132), (124), (142), (134), (143), (234), (243); & (8)\\
(4):&\quad (1234), (1243), (1324), (1342), (1423), (1432); & (9)\\
(2,2):&\quad (12)(34), (13)(24), (14)(23). & (10)
\end{aligned}$$

Question (6)**.** Let G be a finite group with $|G| = n$. Let k be an integer relatively prime to n. Show that the map $x \mapsto x^k, \forall x \in G$, is surjective.

Solution. Since $(k, n) = 1$ there exist $a, b \in \mathbb{Z}$ such that

$$ak + bn = 1. \tag{11}$$

So,

$$x = x^{ak+bn} = x^{ak} x^{bn} = (x^a)^k \cdot (x^b)^n = (x^a)^k, \tag{12}$$

which means the map is surjective.

Question (7)**.** Let G be a group. Let N be the subgroup of G generated by all the elements of the form $xyx^{-1}y^{-1}, \forall x, y \in G$.

(a) Show that if $\phi : G \to G$ is any automorphism of G then $\phi(N) = N$. Conclude that N is a normal subgroup of G.

(b) Show that G/N is abelian.

Solution. (a) For $x, y \in G$ if $\phi(x) = x', \phi(y) = y'$. We have

$$\phi(xyx^{-1}y^{-1}) = \phi(x)\phi(y)\phi(x^{-1})\phi(y^{-1}) = \phi(x)\phi(y)\phi^{-1}(x)\phi^{-1}(y) = x'y'(x')^{-1}(y')^{-1} \in N. \quad (13)$$

That is $\phi(N) \subset N$.

On the other hand for $\forall h, k \in G$ there exists $h', k' \in G$ such that $\phi(h') = h, \phi(k') = k$. So,

$$\phi(h'k'(h')^{-1}(k')^{-1}) = \phi(h')\phi(k')\phi((h')^{-1})\phi((k')^{-1}) = hkh^{-1}k^{-1}. \quad (14)$$

Since $h'k'(h')^{-1}(k')^{-1} \in N$, we have $\phi(N) \supset N$.

Hence, $\phi(N) = N$.

For $g \in G$, we have

$$\begin{aligned} &gxyx^{-1}y^{-1}g^{-1} = gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ =&(gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} \in N. \end{aligned} \quad (15)$$

So, N is a normal subgroup of G.

(b) For $x, y \in G$, we have

$$xNyN = xyN = xyy^{-1}x^{-1}yxN = yxN = yNxN. \quad (16)$$

So, G/N is abelian.