MATH 1050, Homework 6

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October 3, 2013

Due date: Wednesday October 9, 2013

Problem 1: Let v, w be two vertices of a graph G. Show that if there is a walk between v and w then there is a path between v and w. (Hint: take the shortest walk between these vertices.)

Problem 2: (a) Draw Turan's graph $T(7, 2)$. (b) Compute the number of edges in Turan's graph $T(17, 5)$. Explain your answer. (c) (Bonus) Write a formula for the number of edges in Turan's graph $T(n, r)$, where n is not necessarily divisible by r.

Problem 3: Let A be the adjacency matrix of a simple graph G. Show that $\mathrm{tr}(A^3)/6$ is equal to the number of triangles in G. Here tr denotes the trace of a square matrix, and by a triangle in G we mean a subgraph which is a complete graph K_3, in other words a clique of size 3.

Problem 4: Suppose we want to color the vertices of Turan's graph $T(16, 3)$ such that any two vertices which are adjacent (connected by an edge) have different colors. What is the minimum number of colors needed? Prove you answer. (The minimum number of colors needed to color the vertices of a graph G so that no two adjacent vertices share the same color is called the *chromatic number of the graph* G, it is an important concept in graph theory with many applications.)

Problem 1 There is a walk with shortest length

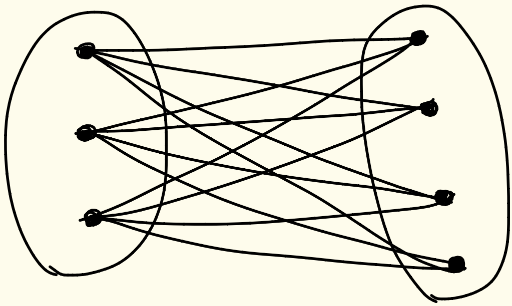
(i.e. with min. number of edges, well-ordering Principle).

Let $v - v_1 - \cdots - v_r - w$ be a shortest walk between v and w. Suppose by contradiction that this is not a path, thus there is a vertex repeated in the walk, e.g. suppose $v_i = v_j$ for $1 \leqslant i < j \leqslant r$. Then $v - \cdots - v_i - v_{j+1} - \cdots - w$ is a walk with (strictly) shorter length between v and w.

The contradiction proves that the shortest walk should be a path and we are done.

Problem 2

(a) $7 = (3\cdot 2)+1 \rightsquigarrow 7 = 3+4$



(b) $17 = 3\times 5 + 2 \rightsquigarrow 17 = 3+3+3+4+4$

Each vertex in a 3-vertex partition is adj. to $14 = 17-3$ other vertices, & each vertex in a 4-vertex partition is adj. to $13 = 17-4$ other vertices. So in total we have:

3 partitions of size 3   2 partitions of size 2

$$\frac{3\times(3\times 14) + 2\times(4\times 13)}{2} = 115$$

each edge counted twice

(c) Let $r_0 = n \bmod r$ and $q = \left[\frac{n}{r}\right]$,

that is, $n = qr + r_0$.

Then there are r_0 partitions of size $q+1$ and $(r-r_0)$ partitions of size q. Each vertex has degree $n-(q+1)$ or $n-q$ depending on what partition it belongs to.

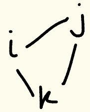
Similar to (b) we get:

$$\#\text{ of edges} = \frac{(r-r_0)q(n-q) + r_0(q+1)(n-q-1)}{2}.$$

(One can verify that this is $\leq \left(1-\frac{1}{r}\right)\frac{n^2}{2}$.)

Problem 3 We know that the entry (i,j) in the matrix A^ℓ is the # of walks of length ℓ from vertex i to vertex j (Thm. in class). Now entry (i,i) in A^3 is the number of walks of length 3 from i to i, &

hence $tr(A^3)$ is the # of walks of length 3 in A.

But each triangle  is counted exactly <u>6</u> times (graph is simple and hence no multiple edges or loops):

$i \to j \to k$ $\quad$ $i \to k \to j$ $\quad$ $j \to i \to k$ $\quad$ $j \to k \to i$

$k \to i \to j$ $\quad$ $k \to j \to i$. Hence $\frac{1}{6} tr(A^3)$ is the number of triangles.

<u>Problem 4</u> We need 3 Colors. If we color each partition of $T(16,3)$ by a different Color then any two adj. vertices get different colors. We Claim that this is not possible with <u>2</u> colors because if we color 16 vertices with <u>2</u> Colors then there should be at least 8 vertices with the same Color. But each partition has at most 6 vertices and hence there

number of partitions

should be two vertices of the same color in different partitions. This is a contradiction because all vertices in different partitions are connected.

In fact, a similar argument proves that the chromatic number of $T(n,r)$ is r.