HW6 Solutions   MATH1250

#4

$\sqrt{3} \notin \mathbb{Q}(\sqrt[3]{2})$ because $\mathbb{Q}(\sqrt{3})$ has deg. 2 over $\mathbb{Q}$ while $\mathbb{Q}(\sqrt[3]{2})$ has deg. 3 and $2 \nmid 3$. Thus $[\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}) : \mathbb{Q}(\sqrt{3})]\,[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 3 \cdot 2 = 6$. Moreover $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$

is a basis for $\mathbb{Q}(\sqrt{3}, \sqrt[3]{2})$ over $\mathbb{Q}(\sqrt{3})$ and $\{1, \sqrt{3}\}$ basis for $\mathbb{Q}(\sqrt{3})$ over $\mathbb{Q}$. It follows that $\{1, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{3}, \sqrt{3}\sqrt[3]{2}, \sqrt{3}\sqrt[3]{4}\}$ is a basis for $\mathbb{Q}(\sqrt{3}, \sqrt[3]{2})$ over $\mathbb{Q}$.

#22

If $b \neq 0$ then $a+bi \notin \mathbb{R}$. By Thm. 31.3, $a+bi$ is alg. over $\mathbb{R}$. Then by Thm. 31.4: $2 = [\mathbb{C} : \mathbb{R}] = [\mathbb{C} : \mathbb{R}(a+bi)]\,[\mathbb{R}(a+bi) : \mathbb{R}]$. and because $a+bi \notin \mathbb{R} \Rightarrow \mathbb{R}(a+bi) \neq \mathbb{R} \Rightarrow [\mathbb{R}(a+bi) : \mathbb{R}] \neq 1$.

(Recall $[E:F] = 1 \Leftrightarrow E = F$). Thus $[\mathbb{R}(a+bi) : \mathbb{R}] = 2 \Rightarrow [\mathbb{C} : \mathbb{R}(a+bi)] = 1$ $\Rightarrow \mathbb{C} = \mathbb{R}(a+bi)$.

#24

If $x^2 - 3$ was red. over $\mathbb{Q}(\sqrt[3]{2})$ then it would factor into linear factors and hence its roots $\pm\sqrt{3}$ would lie in $\mathbb{Q}(\sqrt[3]{2})$.

But we already saw (Ex. 4) that $\sqrt{3} \notin \mathbb{Q}(\sqrt[3]{2})$.

#28

If $a=b$ the result is clear. So let us assume $a \neq b$. It is obvious that $\mathbb{Q}(\sqrt{a}+\sqrt{b}) \subseteq \mathbb{Q}(\sqrt{a}, \sqrt{b})$. We need to prove the reverse inclusion. Let $\alpha = \dfrac{a-b}{\sqrt{a}+\sqrt{b}} \in \mathbb{Q}(\sqrt{a}+\sqrt{b})$.

Note $\alpha = \sqrt{a} - \sqrt{b} \Rightarrow \frac{1}{2}(\alpha + (\sqrt{a}+\sqrt{b})) = \sqrt{a}$ is contained in $\mathbb{Q}(\sqrt{a}+\sqrt{b})$. It follows that $(\sqrt{a}+\sqrt{b}) - \sqrt{a} = \sqrt{b}$ is also in $\mathbb{Q}(\sqrt{a}+\sqrt{b})$. Thus $\mathbb{Q}(\sqrt{a},\sqrt{b}) = \mathbb{Q}(\sqrt{a}+\sqrt{b})$.

#35

If $\operatorname{char} F = p$ is odd then $1 \neq -1$. Because for any $a \in F$, $a^2 = (-a)^2$, the squares of elements of F can run through at most $\frac{|F|+1}{2}$ elements of F, i.e. $a \mapsto a^2$ cannot be an onto function from F to F, thus there are elements which are not square. i.e. $\exists b \in F$ s.t. $x^2 - b$ does not have root in F. Thus F is not alg. closed.

(due to Piotr Przewinski)

Alternative solution. Let $a_1, \ldots, a_m$ be all the elements of F. Consider $f(x) = \left(\prod_{i=1}^{m}(x-a_i)\right) + 1$. Then $f(x)$ is not div. by any $x - a_i$ and hence has no root in F.